

ON WELL-POSEDNESS, STABILITY, AND BIFURCATION FOR THE AXISYMMETRIC SURFACE DIFFUSION FLOW

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ABSTRACT. In this article, we study the axisymmetric surface diffusion flow (ASD), a fourth-order geometric evolution law. In particular, we prove that ASD generates a real analytic semiflow in the space of $(2+\alpha)$ -little-Hölder regular surfaces of revolution embedded in \mathbb{R}^3 and satisfying periodic boundary conditions. We also give conditions for global existence of solutions and prove that solutions are real analytic in time and space. Further, we investigate the geometric properties of solutions to ASD. Utilizing a connection to axisymmetric surfaces with constant mean curvature, we characterize the equilibria of ASD. Then, focusing on the family of cylinders, we establish results regarding stability, instability and bifurcation behavior, with the radius acting as a bifurcation parameter for the problem.

1. INTRODUCTION

The central focus of this article is the development of an analytic setting and rigorous results for the axisymmetric surface diffusion flow (ASD) with periodic boundary conditions. We establish well-posedness of ASD and investigate geometric properties of solutions, including characterizing equilibria and establishing their stability, instability and bifurcation behavior. We establish and take full advantage of *maximal regularity* for ASD. Most notably, with maximal regularity we gain access to the implicit function theorem, a very powerful tool in nonlinear analysis and dynamical systems theory. We begin with a motivation and derivation of the general surface diffusion flow, of which ASD is a special case, and we introduce the general outline of the paper.

The mathematical equations modeling surface diffusion go back to a paper by Mullins [35] from the 1950s, who was in turn motivated by earlier work of Herring [24]. Both of these authors investigate phenomena witnessed in sintering processes, a method by which objects are created by heating powdered material to a high temperature, while remaining below the boiling point of the particular substance. When the applied temperature reaches a critical point, the atoms on the surfaces of individual particles will diffuse across to other particles, fusing the powder together into one solid object. In response to gradients of the chemical potential along the surface of this newly formed object, the surface atoms may undergo diffusive mass transport *on* the surface of the object, attempting to reduce the surface free energy. Given the right conditions – temperature, pressure, grain size, sample size, etc. – the mass flux due to this chemical potential will dominate the dynamics on the

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surface, and it is the resulting morphological evolution of the surface which the surface diffusion flow aims to model. We also note that the surface diffusion flow has been used to model the motion of surfaces in other physical processes (e.g. growth of crystals and nano-structures). The article [8] contains the formulation of the model which we present below, which is set in a more general framework than the original model developed by Mullins.

1.1. The Surface Diffusion Flow. From a mathematical perspective, the governing equation for motion via surface diffusion can be expressed for hypersurfaces in arbitrary space dimensions. In particular, let $\Gamma \subset \mathbb{R}^n$ be a closed, compact, immersed, oriented Riemannian manifold with codimension 1. Then we denote by $\mathcal{H} = \mathcal{H}(\Gamma)$ the (normalized) mean curvature on Γ , which is simply the sum of the principle curvatures on the hypersurface, and Δ_Γ denotes the Laplace–Beltrami operator, or *surface Laplacian*, on Γ . The motion of the surface Γ by surface diffusion is then governed by the equation

$$V = \Delta_\Gamma \mathcal{H},$$

where V denotes the normal velocity of the surface Γ . A solution to the surface diffusion problem on the interval $J \subset \mathbb{R}_+$, with $0 \in J$, is a family $\{\Gamma(t) : t \in J\}$ of closed, compact, immersed hypersurfaces in \mathbb{R}^n which satisfy the equation

$$(1.1) \quad \begin{cases} V(\Gamma(t)) = \Delta_{\Gamma(t)} \mathcal{H}(\Gamma(t)), & t \in \dot{J} := J \setminus \{0\}, \\ \Gamma(0) = \Gamma_0, \end{cases}$$

for a given initial hypersurface Γ_0 . It can be shown that solutions to (1.1) are volume-preserving, in the sense that the signed volume of the region Ω *enclosed* by the surface Γ is preserved along solutions. Additionally, (1.1) is surface-area-reducing.

Well-posedness, for any dimension $n \geq 2$, was established by Escher, Mayer and Simonett in [20], where it was also shown that the $(n - 1)$ -dimensional spheres are asymptotically stable equilibria. Meanwhile, Mayer and Simonett [34] demonstrate the existence of initially embedded hypersurfaces which are driven to self-intersection under the surface diffusion flow. Beyond these initial results, the literature lacks general analytic results regarding the behavior of solutions to (1.1), e.g. occurrence of singularities and conditions under which solutions breakdown in finite time. An important feature of the surface diffusion flow that Escher, Mayer and Simonett exploit in order to obtain well-posedness results is the fact that the equation has a quasilinear structure and the linear part of the equation exhibits *maximal regularity* properties on appropriately chosen function spaces. These features will also play an important role in our analysis of ASD, for which we establish a robust theory for local and global well-posedness.

1.2. Axisymmetric Surface Diffusion (ASD). For the remainder of the paper, we will focus our attention on a special case of the surface diffusion flow. Namely, we consider the case of $\Gamma \subset \mathbb{R}^3$ an embedded surface which is symmetric about an axis of rotation (which we take to be the x -axis, without loss of generality) and satisfies prescribed periodic boundary conditions on some fixed interval L of periodicity (we take $L = [-\pi, \pi]$ and enforce 2π periodicity, without significant loss of generality). In particular, the axisymmetric surface Γ is characterized by the

parametrization

$$\Gamma = \left\{ (x, r(x) \cos(\theta), r(x) \sin(\theta)) : x \in \mathbb{R}, \theta \in [-\pi, \pi] \right\},$$

where the function $r : \mathbb{R} \rightarrow (0, \infty)$ is the *profile function* for the surface Γ . Conversely, a profile function $r : \mathbb{R} \rightarrow (0, \infty)$ generates an axisymmetric surface $\Gamma = \Gamma(r)$ via the parametrization given above.

Utilizing the explicit parametrization for axisymmetric surfaces, we can recast the surface diffusion problem as an evolution equation for the profile functions $r = r(t)$. In particular, one can see that the surface $\Gamma(r)$ inherits the Riemannian metric

$$g = (1 + r_x^2) dx \wedge dx + r^2 d\theta \wedge d\theta,$$

from the embedding $\Gamma \hookrightarrow \mathbb{R}^3$, with respect to the surface coordinates (x, θ) ; where the subscript $f_{x_i} := \partial_{x_i} f$ indicates the derivative of f with respect to the indicated variable x_i . It follows that the (normalized) mean curvature of the surface is $\mathcal{H}(r) = \kappa_1 + \kappa_2$, where

$$\kappa_1 = \frac{1}{r\sqrt{1+r_x^2}} \quad \text{and} \quad \kappa_2 = \frac{-r_{xx}}{(1+r_x^2)^{3/2}}$$

are the *azimuthal* and *axial* principle curvatures, respectively, on $\Gamma(r)$. Meanwhile, the Laplace–Beltrami operator on Γ and the normal velocity of $\Gamma = \Gamma(t)$ are

$$\Delta_{\Gamma(r)} = \frac{1}{r\sqrt{1+r_x^2}} \left(\partial_x \left[\frac{r}{\sqrt{1+r_x^2}} \partial_x \right] + \partial_\theta \left[\frac{\sqrt{1+r_x^2}}{r} \partial_\theta \right] \right),$$

$$V(t) = \frac{r_t}{\sqrt{1+r_x^2}}.$$

Finally, plugging these terms into the equation (1.1) and simplifying, we arrive at the expression

$$(1.2) \quad \begin{cases} r_t = \frac{1}{r} \partial_x \left[\frac{r}{\sqrt{1+r_x^2}} \partial_x \left(\frac{1}{r\sqrt{1+r_x^2}} - \frac{r_{xx}}{(1+r_x^2)^{3/2}} \right) \right], & t > 0, x \in \mathbb{R}, \\ r(t, x + 2\pi) = r(t, x), & t \geq 0, x \in \mathbb{R}, \\ r(0, x) = r_0(x), & x \in \mathbb{R}, \end{cases}$$

for the periodic axisymmetric surface diffusion problem. To simplify notation in the sequel, we define the operator

$$(1.3) \quad G(r) := \frac{1}{r} \partial_x \left[\frac{r}{\sqrt{1+r_x^2}} \partial_x \mathcal{H}(r) \right],$$

which is formally equivalent to the right hand side of the first equation in (1.2).

The first investigations of evolution of an axisymmetric surface via surface diffusion can be traced back to the work of Nichols and Mullins [36, 37] in 1965, where one can already see some of the benefits of this special case of the surface diffusion problem. In particular, Mullins and Nichols are able to take advantage of the symmetry of the problem in order to develop an adequate scheme for numerical techniques. Moreover, Mullins and Nichols are already predicting the finite-time pinch-off of tube like surfaces via surface diffusion flow, a feature similar to the mean curvature flow and a natural phenomenon to study in exactly this axisymmetric setting. Following this seminal work by Nichols and Mullins, there are many publications investigating ASD. Researchers continued to study pinch-off behavior

using numerical methods, c.f. [7, 10, 11, 12, 15, 31, 32], developing schemes for the continuation of solutions after the change of topology that occurs at the moment of pinch-off. Meanwhile, many researchers have focused on the numerical investigation of stability/instability and bifurcation behavior of cylinders, which are natural equilibria of ASD, under perturbations of various types, c.f. [7, 10, 12], though we note that these investigations lack a rigorous analytic framework. For more background on ASD see the paper [7] by Bernoff, Bertozzi, and Witelski, which has served as a motivation for the overall nature and scope of results which have been justified for ASD, via numerical and analytic techniques, which we can hope to establish in our functional-analytic setting.

In Section 2, we prove existence of solutions to (1.2), which, to the best of the authors' knowledge, is the first analytic well-posedness result in the literature for the axisymmetric surface diffusion flow with periodic boundary conditions. In particular, we establish existence and uniqueness of maximal solutions which are analytic in time and space for positive time, with a prescribed singularity at time $t = 0$, for initial conditions which are $(2 + \alpha)$ -little-Hölder continuous in space. Additionally, we establish conditions for global existence and regularity of the semiflow induced by (1.2). With these well-posedness results established, all of which depend heavily upon the theory developed in [29] and the well-posedness results for quasilinear equations with maximal regularity proved by Clément and Simonett [9], we then go on to investigate more general dynamic properties of solutions. In Section 3, we characterize the equilibria of ASD using a result of Delaunay [17] and Kenmotsu [26], which describes all of the constant mean curvature surfaces in the axisymmetric setting. In Section 4, we prove that the family of cylinders with radius $r_\star > 1$ are asymptotically, exponentially stable under a large class of nonlinear perturbations, which maintain the same axis of symmetry and satisfy the prescribed periodic boundary conditions. To prove this stability result, we linearize the equation for ASD and see that the spectrum of the linearized operator is contained in the *left half* of the complex plane, although the spectrum will always contain zero as an eigenvalue. Hence, we reduce the equation, by essentially eliminating non-volume-preserving perturbations, in order to eliminate the zero eigenvalue. Then we use results regarding maximal regularity on exponentially weighted function spaces to generate the desired exponential stability for the reduced equation, which is then transferred back to the (full) ASD problem via a *lifting* operator.

We proceed in the remaining sections with results regarding the instability of cylinders with radius $0 < r_\star < 1$ and the existence of branches of bifurcating equilibria which intersect the family of cylinders at radii $r_\star = 1/\ell$, for every $\ell \in \mathbb{N} := \{1, 2, \dots\}$. The instability result of Section 5 makes use of a contradiction technique reminiscent of results from the theory of ordinary differential equations, c.f. Prüss and Wilke [43]. More precisely, by isolating the linearization of the governing equation, one takes advantage of a *spectral gap* and associated spectral projections in order to derive necessary conditions for *stable* initial data, which in turn lead to a contradiction. We also refer to Prüss, Simonett and Zacher [42] and Prüss, Simonett and Wilke [40] for related results. Finally, in Section 6 we apply classic methods of Crandall and Rabinowitz [13] to establish bifurcation results for ASD. However, even in the reduced setting mentioned above, developed while proving stability results, we find that the eigenvalues associated with the problem are not simple. We restrict our attention to surfaces which are even (symmetric

about the surface $x = 0$) and satisfy prescribed regularity and periodicity, similar to a method used by Escher and Matioc [19]. In this setting, the problem has simple eigenvalues and we are able to derive bifurcation results and apply them back to the full ASD problem via a posteriori symmetries of equilibria. We note that, rather than using the restriction to even functions in order to apply the results of Crandall and Rabinowitz [13], we could also have chosen to apply more general bifurcation techniques, such as the methods contained in the manuscript of Kielhöfer [27, Section I.19], in order to generate (higher dimensional) bifurcation results.

1.3. Maximal Regularity. One tool that we will use extensively throughout the paper is the property of (continuous) maximal regularity, also called *optimal regularity* in the literature. Maximal regularity has received a lot of attention in connection with parabolic partial differential equations and evolution laws, c.f. [2, 3, 4, 5, 9, 28, 30, 39, 41, 44]. Although maximal regularity can be developed in a more general setting, we will focus on the setting of *continuous* maximal regularity and direct the interested reader to the references [2, 30] for a general development of the theory.

Let $\mu \in (0, 1]$, $J := [0, T]$, for some $T > 0$, and let E be a (real or complex) Banach space. Following the notation of [9], we define spaces of continuous functions on $\dot{J} := J \setminus \{0\}$ with prescribed singularity at 0. Namely, define

$$(1.4) \quad \begin{aligned} BUC_{1-\mu}(J, E) &:= \left\{ u \in C(\dot{J}, E) : [t \mapsto t^{1-\mu}u(t)] \in BUC(\dot{J}, E) \text{ and} \right. \\ &\quad \left. \lim_{t \rightarrow 0^+} t^{1-\mu}\|u(t)\|_E = 0 \right\}, \quad \mu \in (0, 1) \\ \|u\|_{B_{1-\mu}} &:= \sup_{t \in J} t^{1-\mu}\|u(t)\|_E, \end{aligned}$$

where BUC denotes the space consisting of bounded, uniformly continuous functions. It is easy to verify that $BUC_{1-\mu}(J, E)$ is a Banach space when equipped with the norm $\|\cdot\|_{B_{1-\mu}}$. Moreover, we define the subspace

$$BUC_{1-\mu}^1(J, E) := \left\{ u \in C^1(\dot{J}, E) : u, \dot{u} \in BUC_{1-\mu}(J, E) \right\}, \quad \mu \in (0, 1)$$

and we set

$$BUC_0(J, E) := BUC(J, E) \quad BUC_0^1(J, E) := BUC^1(J, E).$$

Now, if E_1 and E_0 are a pair of Banach spaces such that E_1 is continuously embedded in E_0 , denoted $E_1 \hookrightarrow E_0$, we set

$$\begin{aligned} \mathbb{E}_0(J) &:= BUC_{1-\mu}(J, E_0), \quad \mu \in (0, 1], \\ \mathbb{E}_1(J) &:= BUC_{1-\mu}^1(J, E_0) \cap BUC_{1-\mu}(J, E_1), \end{aligned}$$

where $\mathbb{E}_1(J)$ is a Banach space with the norm

$$\|u\|_{\mathbb{E}_1(J)} := \sup_{t \in J} t^{1-\mu} \left(\|\dot{u}(t)\|_{E_0} + \|u(t)\|_{E_1} \right).$$

It follows that the trace operator $\gamma : \mathbb{E}_1(J) \rightarrow E_0$, defined by $\gamma v := v(0)$, is well-defined and we denote by $\gamma \mathbb{E}_1$ the image of γ in E_0 , which is itself a Banach space when equipped with the norm

$$\|x\|_{\gamma \mathbb{E}_1} := \inf \left\{ \|v\|_{\mathbb{E}_1(J)} : v \in \mathbb{E}_1(J) \text{ and } \gamma v = x \right\}.$$

For a bounded linear operator $B \in \mathcal{L}(E_1, E_0)$ which is closed as an operator on E_0 , we say $(\mathbb{E}_0(J), \mathbb{E}_1(J))$ is a *pair of maximal regularity* for B and write $B \in \mathcal{M}_\mu(E_1, E_0)$, if

$$\left(\frac{d}{dt} + B, \gamma \right) \in \mathcal{L}_{isom}(\mathbb{E}_1(J), \mathbb{E}_0(J) \times \gamma \mathbb{E}_1),$$

where \mathcal{L}_{isom} denotes the space of bounded linear isomorphisms. In particular, $(\mathbb{E}_0(J), \mathbb{E}_1(J))$ is a pair of maximal regularity for B if and only if for every $(f, u_0) \in \mathbb{E}_0(J) \times \gamma \mathbb{E}_1$, there exists a unique solution $u \in \mathbb{E}_1(J)$ to the inhomogeneous Cauchy problem

$$\begin{cases} \dot{u}(t) + Bu(t) = f(t), & t \in J, \\ u(0) = u_0. \end{cases}$$

Moreover, in the current setting, it follows that $\gamma \mathbb{E}_1 \doteq (E_0, E_1)_{\mu, \infty}^0$, i.e. the trace space $\gamma \mathbb{E}_1$ is topologically equivalent to the noted continuous interpolation spaces of Da Prato and Grisvard, c.f. [2, 9, 14, 30].

2. WELL-POSEDNESS OF (1.2)

When considering the surface diffusion problem, the underlying Banach spaces E_0 and E_1 in the formulation of maximal regularity will be spacial regularity classes which describe the properties of the profile functions $r(t)$. We proceed by defining these regularity classes. We define the one-dimensional torus $\mathbb{T} := [-\pi, \pi]$, where the points $-\pi$ and π are identified, which is equipped with the topology generated by the metric

$$d_{\mathbb{T}}(x, y) := \min\{|x - y|, 2\pi - |x - y|\}, \quad x, y \in \mathbb{T}.$$

There is a natural equivalence between functions defined on \mathbb{T} and 2π -periodic functions on \mathbb{R} which preserves properties of (Hölder) continuity and differentiability. In particular, we will be working with the so-called periodic little-Hölder spaces $h^\sigma(\mathbb{T})$, for $\sigma \in \mathbb{R}_+ \setminus \mathbb{Z}$. Definitions and basic properties of periodic little-Hölder spaces, as well as details on the connection between spaces of functions on \mathbb{T} and 2π -periodic functions on \mathbb{R} can be found in [29] and the references therein. For the readers convenience, we provide a brief definition of $h^\sigma(\mathbb{T})$ below.

For $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, denote by $C^k(\mathbb{T})$ the Banach space of k -times continuously differentiable functions $f : \mathbb{T} \rightarrow \mathbb{R}$, equipped with the norm

$$\|f\|_{C^k(\mathbb{T})} := \sum_{j=0}^k \|f^{(j)}\|_{C(\mathbb{T})} := \sum_{j=0}^k \left(\sup_{x \in \mathbb{T}} |f^{(j)}(x)| \right).$$

Moreover, for $\alpha \in (0, 1)$ and $k \in \mathbb{N}_0$, we define the space $C^{k+\alpha}(\mathbb{T})$ to be those functions $f \in C^k(\mathbb{T})$ such that the α -Hölder seminorm

$$[f^{(k)}]_{\alpha, \mathbb{T}} := \sup_{\substack{x, y \in \mathbb{T} \\ x \neq y}} \frac{|f^{(k)}(x) - f^{(k)}(y)|}{d_{\mathbb{T}}^\alpha(x, y)}$$

is finite. It follows that $C^{k+\alpha}(\mathbb{T})$ is a Banach space when equipped with the norm

$$\|f\|_{C^{k+\alpha}(\mathbb{T})} := \|f\|_{C^k(\mathbb{T})} + [f^{(k)}]_{\alpha, \mathbb{T}}.$$

Finally, we define the periodic little-Hölder space

$$h^{k+\alpha}(\mathbb{T}) := \left\{ f \in C^{k+\alpha}(\mathbb{T}) : \lim_{\delta \rightarrow 0} \sup_{\substack{x, y \in \mathbb{T} \\ 0 < d_{\mathbb{T}}(x, y) < \delta}} \frac{|f^{(k)}(x) - f^{(k)}(y)|}{d_{\mathbb{T}}^{\alpha}(x, y)} = 0 \right\},$$

for $k \in \mathbb{N}_0$ and $\alpha \in (0, 1)$ which is a Banach algebra with pointwise multiplication of functions and equipped with the norm $\|\cdot\|_{h^{k+\alpha}} := \|\cdot\|_{C^{k+\alpha}(\mathbb{T})}$ inherited from $C^{k+\alpha}(\mathbb{T})$. For equivalent definitions and more properties of the periodic little-Hölder spaces, see [29, Section 1].

In order to make explicit the quasilinear structure of (1.2), we reformulate the problem. By expanding the governing equation we arrive at the formally equivalent problem

$$(2.1) \quad \begin{cases} \partial_t r(t, x) + [\mathcal{A}(r(t))r(t)](x) = f(r(t, x)), & t > 0, x \in \mathbb{T}, \\ r(0, x) = r_0(x), & x \in \mathbb{T}, \end{cases}$$

where, for appropriately chosen functions ρ ,

$$(2.2) \quad \mathcal{A}(\rho) := \frac{1}{(1 + \rho_x^2)^2} \partial_x^4 + \frac{2\rho_x(1 + \rho_x^2 - 3\rho\rho_{xx})}{\rho(1 + \rho_x^2)^3} \partial_x^3$$

is a fourth-order differential operator with variable coefficients over \mathbb{T} and

$$(2.3) \quad f(\rho) := \frac{\rho_x^2 - 1}{\rho^2(1 + \rho_x^2)^2} \rho_{xx} + \frac{6\rho_x^2 - 1}{\rho(1 + \rho_x^2)^3} \rho_{xx}^2 + \frac{3 - 15\rho_x^2}{(1 + \rho_x^2)^4} \rho_{xx}^3 + \frac{\rho_x^2}{\rho^3(1 + \rho_x^2)}$$

is a \mathbb{R} -valued function over \mathbb{T} . Looking at these formal expressions, one can deduce several properties that the functions ρ must satisfy in order to get good mapping properties for f and \mathcal{A} . In particular, we want to choose ρ such that $\rho(x) \neq 0$ for all $x \in \mathbb{T}$, also we want that the spacial derivatives ρ_x and ρ_{xx} make sense and the products ρ^2 , ρ^3 , $\rho\rho_x^2$, etc. have desired regularity properties. With these conditions in mind, we proceed with our well-posedness result.

2.1. Existence and Uniqueness of Solutions. Fix $\alpha \in (0, 1)$ and define the spaces of \mathbb{R} -valued little-Hölder continuous functions

$$(2.4) \quad E_0 := h^{\alpha}(\mathbb{T}), \quad E_1 := h^{4+\alpha}(\mathbb{T}), \quad \text{and} \quad E_{\mu} := (E_0, E_1)_{\mu, \infty}^0,$$

where $(\cdot, \cdot)_{\mu, \infty}^0$, for $\mu \in (0, 1)$, denotes the continuous interpolation functor of Da Prato and Grisvard, c.f. [14] or [2]. It is well-known that the little-Hölder spaces are stable under this interpolation method, in particular we know that

$$E_{\mu} = h^{4\mu+\alpha}(\mathbb{T}) \quad (\text{up to equivalent norms}), \quad \text{for} \quad 4\mu + \alpha \notin \mathbb{Z},$$

c.f. [29, 30]. Further, let V be the set of functions $r : \mathbb{T} \rightarrow \mathbb{R}$ such that $r(x) > 0$ for all $x \in \mathbb{T}$ and define $V_{\mu} := V \cap E_{\mu}$ for $\mu \in [0, 1]$. We note that V_{μ} is an open subset of E_{μ} for all $\mu \in [0, 1]$.

Lemma 2.1. *Let $\mu \in [1/2, 1]$. Then*

$$(\mathcal{A}, f) \in C^{\omega} \left(V_{\mu}, \mathcal{MR}_{\nu}(E_1, E_0) \times E_0 \right), \quad \text{for} \quad \nu \in (0, 1],$$

where C^{ω} denotes the space of real analytic mappings between Banach spaces.

Proof. Fix $\mu \in [1/2, 1]$ as indicated. This result relies on the fact that the little-Hölder spaces $h^\sigma(\mathbb{T})$ are Banach algebras with pointwise multiplication of functions.

CLAIM 1: $\mathcal{A}(\rho) \in \mathcal{MR}_\nu(E_1, E_0)$ for $\rho \in V_\mu$, $\nu \in (0, 1]$. This claim will follow from [29], though the setting of that paper differs slightly from the current setting and warrants some discussion. First, for $\rho \in V_\mu$ define the coefficients

$$b_4(\rho) := \frac{1}{(1 + \rho_x^2)^2} \quad \text{and} \quad b_3(\rho) := \frac{2\rho_x(1 + \rho_x^2 - 3\rho\rho_{xx})}{\rho(1 + \rho_x^2)^3},$$

so that $\mathcal{A}(\rho) = b_4(\rho)\partial_x^4 + b_3(\rho)\partial_x^3$. By our choice of μ , it follows that $V_\mu \subset h^{2+\alpha}(\mathbb{T}, \mathbb{R})$, so that $b_4, b_3 \in E_0$ and $\mathcal{A}(\rho)$ is a uniformly elliptic differential operator. By [29, Theorem 5.2] we conclude that

$$\mathcal{A}(\rho) \in \mathcal{MR}_\nu(h^{4+\alpha}(\mathbb{T}, \mathbb{C}), h^\alpha(\mathbb{T}, \mathbb{C})), \quad \nu \in (0, 1],$$

where we utilize the notation $h^{k+\alpha}(\mathbb{T}, \mathbb{C})$ to be clear that the space consists of \mathbb{C} -valued functions over \mathbb{T} , and does not coincide with the spaces E_μ being considered herein. However, $h^{k+\alpha}(\mathbb{T}, \mathbb{C})$ does coincide with the complexification of $h^{k+\alpha}(\mathbb{T}, \mathbb{R})$ (up to equivalent norms) and it is a straightforward exercise to see that the property of maximal regularity continues to hold under restriction to the subspaces $h^\sigma(\mathbb{T}, \mathbb{R})$.

CLAIM 2: The operation of inversion is real analytic from V_0 into E_0 , i.e.

$$T_i : [r \mapsto 1/r] \in C^\omega(V_0, E_0).$$

Fix $r_0 \in V_0$ and choose $a > 0$ so that $r_0(x) > a$ for all $x \in \mathbb{T}$. If $r \in E_0$ is chosen so that $\|r - r_0\|_{C(\mathbb{T})} < a$, then the representation

$$\frac{1}{r(x)} = \frac{1}{r_0(x) \left(1 + \frac{r(x) - r_0(x)}{r_0(x)}\right)} = \frac{1}{r_0(x)} \sum_{n=0}^{\infty} \left(\frac{r_0(x) - r(x)}{r_0(x)}\right)^n,$$

holds for $x \in \mathbb{T}$ arbitrary, where the last equality follows by an elementary geometric series argument. Hence, the given power series represents the function $1/r$ pointwise for $x \in \mathbb{T}$. Moreover, it follows from the algebraic structure of E_0 that

$$S_k := \sum_{n=0}^k \frac{(r_0 - r)^n}{r_0^{n+1}} \in E_0, \quad \text{for } k \in \mathbb{N}.$$

Finally, if $\|r_0 - r\|_{E_0} \leq a/2$, then

$$\sum_{n=0}^{\infty} \left\| \frac{(r_0 - r)^n}{r_0^{n+1}} \right\|_{E_0} \leq \sum_{n=0}^{\infty} \frac{\|r_0 - r\|_{E_0}^n}{\|r_0\|_{E_0}^{n+1}} \leq \sum_{n=0}^{\infty} \frac{(a/2)^n}{a^{n+1}} = 2/a,$$

which demonstrates the power series converges absolutely in the topology of E_0 and the claim follows.

CLAIM 3: The operations of differentiation and multiplication are real analytic in the setting of little-Hölder spaces, i.e.

$$\begin{aligned} \partial_x : [r \mapsto r_x] &\in C^\omega(h^{\sigma+1}(\mathbb{T}), h^\sigma(\mathbb{T})), \quad \sigma \in \mathbb{R}_+ \setminus \mathbb{Z}, \\ T_m : [(r, s) \mapsto rs] &\in C^\omega(E_0 \times E_0, E_0). \end{aligned}$$

This claim follows since ∂_x and T_m are bounded, linear and bilinear (respectively) on the indicated spaces.

The regularity of the mappings \mathcal{A} and f now follows from the preceding claims, with the additional observation that the mapping $\mathcal{A} : V_\mu \rightarrow \mathcal{L}(E_1, E_0)$ inherits the

regularity of the coefficients $b_3, b_4 : V_\mu \rightarrow E_0$ and the fact that $\mathcal{MR}_\mu(E_1, E_0)$ is an open subset of $\mathcal{L}(E_0, E_1)$, c.f. [9, Lemma 2.5(a)]. \square

With Lemma 2.1 established, we can take full advantage of the well-posedness results for quasilinear parabolic equations presented in the article [9] of Clément and Simonett. In particular, we conclude results regarding local existence and uniqueness of solutions with continuous dependence on initial data, as well as maximal solutions and conditions for global existence. However, we give only a limited presentation, focusing on those results which will be of most direct use to us in the sequel, and we refer the interested reader to [9] for further details on well-posedness of (2.1).

Before we can properly state a result on maximal solutions, we need to introduce one more space of functions from an interval $J \subset \mathbb{R}_+$ to a Banach space E , with prescribed singularity at zero. Namely, if $J = [0, a)$ for $a > 0$, i.e. J is a right-open interval containing 0, then we set

$$\begin{aligned} C_{1-\mu}(J, E) &:= \{u \in C(\dot{J}, E) : u \in BUC_{1-\mu}([0, T], E), \quad T < \sup J\}, \\ C_{1-\mu}^1(J, E) &:= \{u \in C^1(\dot{J}, E) : u, \dot{u} \in C_{1-\mu}(J, E)\}, \quad \mu \in (0, 1], \end{aligned}$$

which we equip with the natural Fréchet topologies induced by $BUC_{1-\mu}([0, T], E)$ and $BUC_{1-\mu}^1([0, T], E)$, respectively.

Theorem 2.2 (Existence and Uniqueness). *Fix $\alpha \in (0, 1)$ and take $\mu \in [1/2, 1]$ so that $4\mu + \alpha \notin \mathbb{Z}$. For each initial value $r_0 \in V_\mu := h^{4\mu+\alpha}(\mathbb{T}) \cap [r > 0]$, there exists a unique maximal solution*

$$r(\cdot, r_0) \in C_{1-\mu}^1(J(r_0), h^\alpha(\mathbb{T})) \cap C_{1-\mu}(J(r_0), h^{4+\alpha}(\mathbb{T})),$$

where $J(r_0) = [0, t^+(r_0)) \subseteq \mathbb{R}_+$ denotes the maximal interval of existence for initial data r_0 . Further, it follows that

$$\mathcal{D} := \bigcup_{r_0 \in V_\mu} J(r_0) \times \{r_0\}$$

is open in $\mathbb{R}_+ \times V_\mu$ and $\varphi : [(t, r_0) \mapsto r(t, r_0)]$ is an analytic semiflow on V_μ , i.e. using the notation $\varphi^t(r_0) := \varphi(t, r_0)$, the mapping φ satisfies the conditions

- $\varphi \in C(\mathcal{D}, V_\mu)$
- $\varphi^0 = id_{V_\mu}$
- $\varphi^{s+t}(r_0) = \varphi^t \circ \varphi^s(r_0)$ for $0 \leq s < t^+(r_0)$ and $0 \leq t < t^+(\varphi^s(r_0))$
- $\varphi(t, \cdot) \in C^\omega(\mathcal{D}_t, V_\mu)$ for $t \in \mathbb{R}_+$ with $\mathcal{D}_t := \{r \in V_\mu : (t, r) \in \mathcal{D}\} \neq \emptyset$.

Proof. In case $\mu \in [1/2, 1)$, the result follows from Lemma 2.1 and [9, Theorems 4.1, 5.1 and 6.1]. When $\mu = 1$ we note that the existence and uniqueness of a maximal solution

$$r(\cdot, r_0) \in C^1(J(r_0), E_0) \cap C(J(r_0), E_1)$$

follows from [9, Theorem 4.1(b)]. However, for the semiflow properties, we will consider (1.2) as a fully nonlinear equation, and apply results of Angenent [5]. In particular, for $r \in V_1$ we use the representation $G(r) = -\mathcal{A}(r)r + f(r)$ and (2.2)–(2.3) to see that the Fréchet derivative DG has the structure

$$DG(r) = -\frac{1}{(1 + r_x^2)^2} \partial_x^4 + \sum_{k=0}^3 B_k(r) \partial_x^k,$$

where the coefficients $B_k(r) \in E_0$ for every $r \in V_1$, $k = 0, \dots, 3$. From this computation it follows that $-DG(r)$ is a uniformly elliptic operator from E_1 to E_0 and so, using the results of [29] as in Claim 1 of Lemma 2.1 above, we see that $-DG(r) \in \mathcal{MR}_1(E_1, E_0)$ for all $r \in V_1$. Now the fact that (1.2) generates an analytic semiflow on V_1 follows from [5, Corollary 2.9]. \square

The results contained in [9] also give the following conditions for global solutions. We have separated this result from the previous existence result because breakdown of solutions to (2.1), in particular an analytic investigation of *pinch-off* behavior of certain solutions, is an open and interesting topic.

Theorem 2.3 (Global Solutions). *Let $r_0 \in V_\mu := h^{4\mu+\alpha}(\mathbb{T}) \cap [r > 0]$ for $\mu \in (1/2, 1]$, such that $4\mu + \alpha \notin \mathbb{Z}$, and suppose there exists $0 < M < \infty$ so that, for all $t \in J(r_0)$*

- $r(t, r_0)(x) \geq 1/M$, $\forall x \in \mathbb{T}$, and
- $\|r(t, r_0)\|_{h^{4\mu+\alpha}(\mathbb{T})} \leq M$,

then it must hold that $t^+(r_0) = \infty$, so that $r(\cdot, r_0)$ is a global solution. Conversely, if $r_0 \in V_\mu$ and $t^+(r_0) < \infty$, i.e. the solution breaks down in finite-time, then one, or both, of the conditions stated must fail to hold.

We can also state the following result regarding analyticity of the maximal solutions $r(\cdot, r_0)$ in both space and time.

Theorem 2.4 (Regularity of Solutions). *Under the same assumptions as in Theorem 2.2, it follows that*

$$(2.6) \quad r(\cdot, r_0) \in C^\omega((0, t^+(r_0)) \times \mathbb{T}) \quad \text{for all } r_0 \in V_\mu, \quad \mu \in [1/2, 1].$$

Proof. Here we rely on an idea that goes back to Masuda [33] and Angenent [5, 6] to introduce parameters and use the implicit function theorem to obtain regularity results for solutions, see also [21, 22, 23].

First, for $a \in \mathbb{R}$ let $T_a : \mathbb{T} \rightarrow \mathbb{T}$ be the translation operator, where $T_a(x)$ denotes the unique element in \mathbb{T} that is in the coset $[x + a] \in \mathbb{R}/2\pi\mathbb{Z}$ of $(x + a)$. T_a naturally acts on functions $u \in C(\mathbb{T}, \mathbb{R})$ by virtue of $(T_a u)(x) := u(T_a(x))$. As in [22] one shows that, for $a \in \mathbb{R}$, the family of translations $\{T_a : a \in \mathbb{R}\}$ induces a strongly continuous group of contractions on any of the spaces E_μ , with infinitesimal generator A_a given by

$$D(A_a) = h^{1+4\mu+\alpha}(\mathbb{T}, \mathbb{R}), \quad A_a = a\partial_x.$$

Let $r_0 \in V_\mu$ be fixed, and let

$$r = r(\cdot, r_0) \in C_{1-\mu}^1(J(r_0), E_0) \cap C_{1-\mu}(J(r_0), E_1)$$

be the unique solution to (2.1) on the maximal interval of existence $J(r_0) = [0, t^+(r_0))$. Let $t_1 \in (0, t^+(r_0))$ be fixed and set $I := [0, t_1]$. Then there exists $\delta > 0$ such that $(1 + \lambda)t \in J(r_0)$ for all $(t, \lambda) \in I \times (-\delta, \delta)$. Finally, for $(\lambda, a) \in W := (-\delta, \delta)^2$ we set

$$r_{\lambda,a}(t) := T_a r((1 + \lambda)t), \quad t \in I;$$

i.e. $r_{\lambda,a}(t, x) = r((1 + \lambda)t, T_a(x))$ for $(t, x) \in I \times \mathbb{T}$. One verifies that

$$r_{\lambda,a} \in \mathbb{E}_1(I) := BUC_{1-\mu}^1(I, E_0) \cap BUC_{1-\mu}(I, E_1).$$

Moreover, since the nonlinear mapping $[r \mapsto G(r)]$ is equivariant with respect to translations, i.e. $T_b G(r) = G(T_b r)$ for any $b \in \mathbb{R}$, we obtain that $r_{\lambda,a}$ is a solution of the parameter-dependent equation

$$(2.7) \quad \begin{cases} \partial_t v = (1 + \lambda)G(v) + a\partial_x v, & t > 0, \\ v(0) = r_0, \end{cases}$$

on the time interval I .

Now, for $\mathbb{U}(I) := \mathbb{E}_1(I) \cap C(I, V)$ we define

$\Phi : \mathbb{U}(I) \times W \rightarrow \mathbb{E}_0(I) \times E_\mu$, $\Phi(v, (\lambda, a)) = (\partial_t v - (1 + \lambda)G(v) - a\partial_x v, \gamma v - r_0)$, where $\mathbb{E}_0(I) := BUC_{1-\mu}(I, E_0)$, and we note that $\Phi(r_{\lambda,a}, (\lambda, a)) = (0, 0)$. Moreover,

$$\Phi \in C^\omega(\mathbb{U}(I) \times W, \mathbb{E}_0(I) \times E_\mu), \quad D_1 \Phi(r, (0, 0)) = \left(\frac{d}{dt} - DG(r), \gamma \right),$$

where we use the same notation for $r = r(\cdot, r_0)$ and its restriction to the time interval I . Exactly as in the proof of [9, Theorem 6.1] one shows that

$$D_1 \Phi(r, (0, 0)) \in \mathcal{L}_{isom}(\mathbb{E}_1(I), \mathbb{E}_0(I) \times E_\mu).$$

Finally, according to the implicit function theorem, c.f. [16, Theorem 15.3] or [18, (10.2.1)], there exist a neighborhood of r in $\mathbb{E}_1(I)$ and a neighborhood of $(0, 0)$ in \mathbb{R}^2 , which we will again denote by $\mathbb{U}(I)$ and W , respectively, and a mapping $g \in C^\omega(W, \mathbb{E}_1(I))$ such that

$$\Phi(v, (\lambda, a)) = (0, 0) \quad \text{if and only if} \quad v = g(\lambda, a)$$

whenever $(v, (\lambda, a)) \in \mathbb{U}(I) \times W$. We conclude that $g(\lambda, a) = r_{\lambda,a}$ and

$$(2.8) \quad [(\lambda, a) \mapsto r_{\lambda,a}] \in C^\omega(W, \mathbb{U}(I)).$$

For $t_0 \in (0, t_1)$ and $x_0 \in \mathbb{T}$ fixed, we see that

$$(2.9) \quad [(\lambda, a) \mapsto r((1 + \lambda)t_0, T_{t_0 a}(x_0))] \in C^\omega(W, \mathbb{R}),$$

and the assertion follows since (t_0, x_0) can be chosen arbitrarily. \square

Remark 2.5. The preceding results can be slightly weakened to allow for arbitrary values of $\mu \in (1/21]$, i.e. without eliminating the possibility that $4\mu + \alpha \in \mathbb{Z}$, by taking initial data from the continuous interpolation spaces $(E_0, E_1)_{\mu, \infty}^0$, which coincide with the Zygmund spaces over \mathbb{T} .

3. CHARACTERIZING THE EQUILIBRIA OF ASD

With well-posedness of (1.2) established, we move on to investigate geometric properties of solutions. We begin our analysis of the long-time behavior of solutions by characterizing and describing the equilibria of (1.2). For this characterization, we make use of a well-known, strict Lyapunov functional for the surface diffusion flow, namely the surface area functional, and a characterization of surfaces of revolution with prescribed mean curvature, as presented by Kenmotsu [26].

Recalling the operator G , as expressed by (1.3) and taking it to be defined on $V_1 \subset h^{4+\alpha}(\mathbb{T})$, one will see that the set of equilibria of (1.2) coincides with the null set of G . Although, from the well-posedness results of the previous section, we know that we can consider (1.2) with initial conditions in $h^{2+\alpha}(\mathbb{T})$, upon which the operator G is not defined, one immediately sees that all equilibria must be in $h^{4+\alpha}(\mathbb{T})$ (in fact, by Theorem 2.4, we can even conclude that equilibria are in

$C^\infty(\mathbb{T})$). More specifically, if we define equilibria to be those elements $\bar{r} \in V_{1/2} = V \cap h^{2+\alpha}(\mathbb{T})$, such that the maximal solution $r(\cdot, \bar{r})$ satisfies

$$r(t, \bar{r}) = \bar{r}, \quad t > 0,$$

then it follows immediately that $\bar{r} \in h^{4+\alpha}(\mathbb{T})$ and $G(\bar{r}) = 0$. Now, we proceed by characterizing the elements of the null set of G .

Consider the functional

$$S(r) := \int_{\mathbb{T}} r(x) \sqrt{1 + r_x^2(x)} dx,$$

which corresponds to the surface area of the generated surface $\Gamma(r)$ and is a strict Lyapunov functional for (4.1). Indeed, if $r = r(\cdot, r_0)$ is a solution to (1.2) on the interval $J(r_0)$, then (suppressing the variable of integration)

$$\begin{aligned} \partial_t S(r(t)) &= \int_{\mathbb{T}} \left[\sqrt{1 + r_x^2(t)} + \frac{r(t)r_x(t)}{\sqrt{1 + r_x^2(t)}} \partial_x \right] G(r(t)) dx \\ &= \int_{\mathbb{T}} \partial_x \left(\frac{r(t)}{\sqrt{1 + r_x^2(t)}} \partial_x \mathcal{H}(r(t)) \right) \mathcal{H}(r(t)) dx \\ &= - \int_{\mathbb{T}} \frac{r(t)}{\sqrt{1 + r_x^2(t)}} (\partial_x \mathcal{H}(r(t)))^2 dx, \quad t \in J(r_0) \setminus \{0\}, \end{aligned}$$

where we use integration by parts twice and eliminate boundary terms because of periodicity. Notice that the expression is non-positive for all times $t \in J(r_0) \setminus \{0\}$. Moreover, if \bar{r} is an equilibrium of (1.2) it follows that $\partial_x \mathcal{H}(\bar{r})$ is identically zero on \mathbb{T} . Meanwhile, notice by the definition of the operator G that $G(\bar{r}) = 0$ whenever $\partial_x \mathcal{H}(\bar{r}) = 0$. Hence, we conclude that $S(r)$ is a strict Lyapunov functional for (1.2), as claimed, and we also see that the equilibria of (1.2) are exactly those functions $\bar{r} \in h^{4+\alpha}(\mathbb{T})$ for which the mean curvature function $\mathcal{H}(\bar{r})$ is constant on \mathbb{T} .

The axisymmetric surfaces with constant mean curvature have been characterized explicitly by Kenmotsu in [26]. In particular, we see that all equilibria of (1.2) are so-called *undulatory* curves, and the *unduloid* surfaces, which are generated by the undulatory curves by revolution about the axis of symmetry, are stationary solutions of the original surface diffusion problem (1.1).

Theorem 3.1 (Delaunay [17] and Kenmotsu [26]). *Any complete surface of revolution with constant mean curvature \mathcal{H} is either a sphere, a catenoid, or a surface whose profile curve is given (up to translation along the axis of symmetry) by the parametric expression, parametrized by the arc-length parameter $s \in \mathbb{R}$,*

$$(3.1) \quad R(s; \mathcal{H}, B) := \left(\int_{\pi/2\mathcal{H}}^s \frac{1 + B \sin(\mathcal{H}t)}{\sqrt{1 + B^2 + 2B \sin(\mathcal{H}t)}} dt, \frac{\sqrt{1 + B^2 + 2B \sin(\mathcal{H}s)}}{|\mathcal{H}|} \right).$$

Remarks 3.2. We can immediately draw several conclusions from Theorem 3.1 and characterize the equilibria of (1.2). We use the notation $R(\mathcal{H}, B)$ to denote the graph in \mathbb{R}^2 of the parametric expression $R(\cdot; \mathcal{H}, B)$.

- a) Although the curves $R(\mathcal{H}, B)$ are well-defined for arbitrary values $B \in \mathbb{R}$ and $\mathcal{H} \neq 0$, it is not difficult to see that, up to translations along the x -axis, we may restrict our attention to values $\mathcal{H} > 0$ and $B \geq 0$, c.f. [26, Section

- 2]. However, in the sequel we will consider the unduloids in the setting of even functions on \mathbb{T} , for which we will benefit by allowing B to range over all of \mathbb{R} .
- b) When $|B| = 1$, $R(\mathcal{H}, 1)$ corresponds to a family of spheres controlled by the parameter \mathcal{H} . The spheres are a well-known family of stable equilibria for the surface diffusion flow, c.f. [20], however their profile curves are outside of our current setting because they fail to be continuously differentiable functions on all of \mathbb{T} . Moreover, we should note that the spheres represented by $R(\mathcal{H}, \pm 1)$ are in fact a *connected* family of spheres, or a *chain of pearls* (see Figure 1)¹, for which even general techniques for (1.1) break down, as the manifold is singular at the points of intersection. These families of connected spheres may be interesting objects to investigate in a weaker formulation of ASD, but they fall outside of the current setting.
 - c) Catenoids, or more precisely the generating catenary curves (which are essentially just the hyperbolic cosine, up to scaling), do not fall into the current setting because they fail to satisfy the periodic boundary conditions, c.f. Figure 1.
 - d) In case $|B| > 1$, the curve $R(\mathcal{H}, B)$ is called a *nodary* (see Figure 2), which cannot be realized as the graph of a function over the x -axis and hence falls outside the current setting.
 - e) For values $0 \leq |B| < 1$, $R(\mathcal{H}, B)$ is a family of *undulary* curves, which generate the *unduloid* surfaces. The undulary curves are representable as graphs of functions over the x -axis, which are strictly positive for B in the given range (see Figure 3). In fact, the case $B = 0$ corresponds to the cylinder of radius $1/\mathcal{H}$. Hence, by Theorem 3.1 above, we conclude that **all equilibria of (1.2) fall into the family of undulary curves.**
 - f) Notice that the curve $R(\mathcal{H}, B)$ is always periodic in both the parameter s and the spacial variable x . In order to ensure that the curve satisfies the 2π -periodic boundary conditions enforced in (1.2) (which we emphasize is a condition regarding periodicity over the variable x and not the arc-length parameter s), we must impose further conditions on the parameters \mathcal{H} and B ; here we avoid $B = 0$ because the curve $R(\mathcal{H}, 0)$ trivially satisfies periodic boundary conditions. In particular, for $B \neq 0$, if \mathcal{H} and B satisfy the relationship

$$(3.2) \quad \frac{\pi \mathcal{H}}{k} = \int_{\pi/2}^{3\pi/2} \frac{1 + B \sin t}{\sqrt{1 + B^2 + 2B \sin t}} dt,$$

then the curve $R(\mathcal{H}, B)$ is $2\pi/k$ periodic in the x variable, for $k \in \mathbb{N}$. In the sequel, we will use the notation $R(B, k)$ to denote the $2\pi/k$ periodic undulary curve with free parameter $-1 < B < 1$ and parameter $\mathcal{H} = \mathcal{H}(B)$ fixed according to (3.2).

¹All of the figures contained herein were generated with the program GNU Octave, version 3.4.3, copyright 2011 John W. Eaton, and GNUPLOT, version 4.4 patchlevel 3, copyright 2010 Thomas Williams, Colin Kelley.

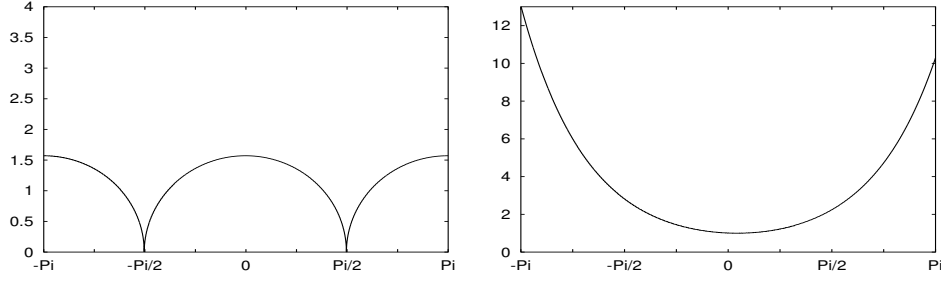
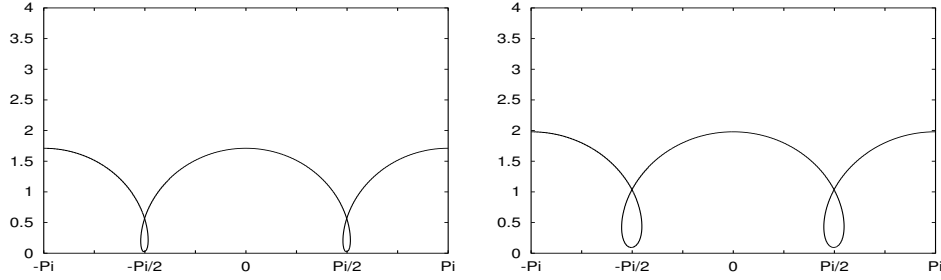
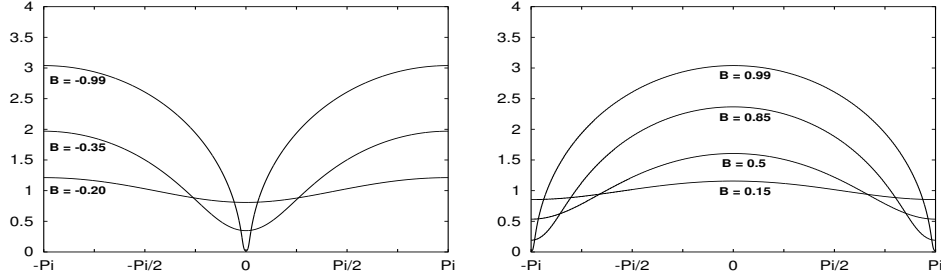


FIGURE 1. Profile curves for a family of spheres and a catenoid, respectively.

FIGURE 2. π periodic nodary curves with $B = 1.03$ and $B = 1.1$, respectively.FIGURE 3. Families of 2π periodic undulary curves with selected parameter values from $B = -0.99$ to $B = 0.99$, as indicated.

4. STABILITY OF CYLINDERS WITH LARGE RADIUS

As seen above, the constant function $r(x) \equiv r_*$, for $r_* > 0$, is an equilibrium of (2.1). Moreover, the constant function $r(x) \equiv r_*$ is associated to the cylinder $\Gamma(r_*)$ with radius r_* , which is a stationary solution of the original surface diffusion problem (1.1). In this section, we establish tools for and carry out the investigation of nonlinear stability for these equilibria.

4.1. Preliminary Analysis and Definitions. Throughout this analysis, we consider an arbitrary $r_* > 0$ and $\sigma \in \mathbb{R}_+ \setminus \mathbb{Z}$, unless otherwise stated. Focusing on

the properties of solutions near r_* , we shift our equations, including the *shifted* operator

$$G_*(\rho) := G(\rho + r_*) = \frac{1}{\rho + r_*} \partial_x \left[\frac{\rho + r_*}{\sqrt{1 + \rho_x^2}} \partial_x \mathcal{H}(\rho + r_*) \right],$$

which maps $\rho \in E_1 \cap U_*$ to E_0 , where we consider $\rho = r - r_*$, and is in the regularity class C^ω by Lemma 2.1; here we take $U_* := V - r_* := \{\rho - r_* : \rho \in V\}$. Now we consider the surface diffusion problem shifted by r_* ,

$$(4.1) \quad \begin{cases} \rho_t(t, x) = G_*(\rho(t, x)), & x \in \mathbb{T}, t > 0, \\ \rho(0, x) = \rho_0(x), & x \in \mathbb{T}, \end{cases}$$

where $\rho_0 := r_0 - r_*$. We say that

$$\rho = \rho(\cdot, \rho_0) \in C^1(\dot{J}, E_0) \cap C(\dot{J}, E_1) \cap C(J, E_\mu \cap U_*)$$

is a solution to (4.1), with initial data $\rho_0 \in E_\mu \cap U_*$, on the interval $J \subset \mathbb{R}_+$ if ρ satisfies (4.1) pointwise, for $t > 0$, and $\rho(0) = \rho_0$. We will investigate the mapping properties of G_* around 0 in order to gain information about the stability of r_* in (1.2).

Define the functional

$$F_*(\rho) = F_*(\rho; r_*) := \int_{\mathbb{T}} (\rho(x) + r_*)^2 dx,$$

which corresponds to the volume enclosed by the surface $\Gamma(\rho + r_*)$. Considering the regularity of F_* , it follows from the analyticity of multiplication and integration on little-Hölder spaces that F_* is of class C^ω from $h^\sigma(\mathbb{T})$ to \mathbb{R} , $\sigma \in \mathbb{R}_+ \setminus \mathbb{Z}$. The Fréchet derivative of F_* is

$$(4.2) \quad DF_*(\rho) : \left[h \mapsto 2 \int_{\mathbb{T}} (\rho(x) + r_*) h(x) dx \right] \in \mathcal{L}(h^\sigma(\mathbb{T}), \mathbb{R}), \quad \rho \in h^\sigma(\mathbb{T}, \mathbb{R}).$$

Moreover, it holds that $F_*(\rho)$ is conserved along solutions to (4.1). Indeed, if $\rho = \rho(\cdot, \rho_0)$ is a solution to (4.1), then

$$\partial_t F_*(\rho(t)) = 2 \int_{\mathbb{T}} (\rho(t) + r_*) \rho_t(t) dx = 2 \int_{\mathbb{T}} \partial_x \left[\frac{(\rho(t) + r_*)}{\sqrt{1 + \rho_x^2(t)}} \partial_x \mathcal{H}(\rho(t) + r_*) \right] dx = 0,$$

for $t \in J(\rho_0) \setminus \{0\}$, where the last equality holds by periodicity. Thus, conservation of F_* along the solution ρ follows by continuity of F_* and convergence of ρ to the initial data ρ_0 in E_μ . From these properties, it follows that

$$(4.3) \quad \mathcal{M}_\eta^\sigma := \left\{ \rho \in h^\sigma(\mathbb{T}) : F_*(\rho) = F_*(\eta) \right\}, \quad \eta \in \mathbb{R}, \sigma \in \mathbb{R}_+ \setminus \mathbb{Z}$$

is a family of invariant level sets for (4.1). The following techniques are motivated by results of Prokert [38] and Vondenhoff [45], whereby one can take advantage of invariant manifolds in order to derive stability results.

First, we introduce the mapping

$$P_0 \rho := \rho - \frac{1}{2\pi} \int_{\mathbb{T}} \rho(x) dx,$$

which defines a projection on $h^\sigma(\mathbb{T})$. We denote by $h_0^\sigma(\mathbb{T})$ the image $P_0(h^\sigma(\mathbb{T}))$, which exactly coincides with the zero-mean functions on \mathbb{T} in the regularity class

$h^\sigma(\mathbb{T})$, and we have the topological decomposition

$$h^\sigma(\mathbb{T}) = h_0^\sigma(\mathbb{T}) \oplus (1 - P_0)(h^\sigma(\mathbb{T})) \cong h_0^\sigma(\mathbb{T}) \oplus \mathbb{R}.$$

In what follows, we equate the constant function $[\eta(x) \equiv \eta] \in (1 - P_0)(h^\sigma(\mathbb{T}))$ with the value $\eta \in \mathbb{R}$, and we denote each simply as η .

Consider the operator

$$\Phi(\rho, \tilde{\rho}, \eta) := \left(P_0\rho - \tilde{\rho}, F_\star(\rho) - F_\star(\eta) \right),$$

which maps $h^\sigma(\mathbb{T}) \times h_0^\sigma(\mathbb{T}) \times \mathbb{R}$ to $h_0^\sigma(\mathbb{T}) \times \mathbb{R}$ and is of class C^ω , by regularity of the mappings F_\star and P_0 . Notice that $\Phi(0, 0, 0) = (0, 0)$ and, using (4.2),

$$(4.4) \quad D_1\Phi(0, 0, 0) = \left(P_0, 4\pi r_\star(1 - P_0) \right) \in \mathcal{L}_{isom}(h^\sigma(\mathbb{T}), h_0^\sigma(\mathbb{T}) \times \mathbb{R}),$$

i.e. the Fréchet derivative of Φ with respect to the first variable, at the origin, is a linear isomorphism. Hence, it follows from the implicit function theorem that there exist neighborhoods $(0, 0) \in U = U_0 \times U_1 \subset h_0^\sigma(\mathbb{T}) \times \mathbb{R}$ and $0 \in U_2 \subset h^\sigma(\mathbb{T})$ and a C^ω function $\psi : U \rightarrow U_2$ such that, for all $(\rho, \tilde{\rho}, \eta) \in U_2 \times U$,

$$\Phi(\rho, \tilde{\rho}, \eta) = (0, 0) \quad \text{if and only if} \quad \rho = \psi(\tilde{\rho}, \eta).$$

Remarks 4.1. We can immediately state the following properties of ψ , which follow directly from its definition and elucidate the relationship between P_0 and ψ .

- a) $P_0\psi(\tilde{\rho}, \eta) = \tilde{\rho}$ for all $(\tilde{\rho}, \eta) \in U$.
- b) Given $\rho \in \psi(U) \cap \mathcal{M}_\eta^\sigma$, it follows that $\psi(P_0\rho, \eta) = \rho$.
- c) $\psi(0, \eta) = \eta$, for $\eta \in U_1$. This and the preceding remark follow from the fact that $F_\star(\eta)$ is injective when restricted to $\eta \in (-r_\star, \infty) \subset \mathbb{R}$.
- d) It follows from the identity $\Phi(\psi(\tilde{\rho}, \eta), \tilde{\rho}, \eta) = (0, 0)$ and differentiating with respect to $\tilde{\rho}$ that $D_1\Phi(\psi(0, \eta), 0, \eta)D_1\psi(0, \eta)h - (h, 0) = (0, 0)$. From this observation, and the fact that $D_1\Phi(\eta, 0, \eta) = (P_0, 4\pi(r_\star + \eta)(1 - P_0))$, it follows that

$$D_1\psi(0, \eta)h = h, \quad h \in h_0^\sigma(\mathbb{T}), \quad \eta \in U_1.$$

- e) $\psi(U_0, \eta) \subset \mathcal{M}_\eta^\sigma$ for $\eta \in U_1$. Hence, $\psi(\cdot, \eta)$ can be taken as a (local) parametrization of \mathcal{M}_η^σ . Moreover, from the preceding remark and the bijectivity of $\psi(\cdot, \eta)$ from U_0 to $\mathcal{M}_\eta^\sigma \cap U_2$, we can see that $\mathcal{M}_\eta^\sigma \cap U_2$ is a Banach manifold over $h_0^\sigma(\mathbb{T})$ anchored at the point $\eta \in \mathbb{R}$.
- f) For $(\tilde{\rho}, \eta) \in U$, we have the representation

$$\psi(\tilde{\rho}, \eta) = \left(P_0 + (1 - P_0) \right) \psi(\tilde{\rho}, \eta) = \tilde{\rho} + \frac{1}{2\pi} \int_{\mathbb{T}} \psi(\tilde{\rho}, \eta)(x) dx,$$

and so we can see that $\mathcal{M}_\eta^\sigma \cap U_2$ can be realized (locally) as the graph of a \mathbb{R} -valued analytic function over the zero-mean functions $\tilde{\rho} \in h_0^\sigma(\mathbb{T})$.

- g) Although $\psi(\cdot, \eta)$ depends upon the parameter σ , a priori, it follows easily from the preceding representation that

$$\psi(\cdot, \eta) : U_0 \cap h_0^{\tilde{\sigma}}(\mathbb{T}) \rightarrow h^{\tilde{\sigma}}(\mathbb{T}), \quad \tilde{\sigma} \in \mathbb{R}_+ \setminus \mathbb{Z},$$

so that ψ preserves the spacial regularity of functions regardless of the regularity parameter σ with which ψ was constructed. However, notice that the neighborhood U_0 will remain intrinsically linked with the parameter which was used to construct ψ .

With the established invariance and local structure of the sets \mathcal{M}_η^σ , it follows that the dynamics governing solutions to (1.2) reside in the tangent space to the manifold $\mathcal{M}_\eta^\sigma \cap U_2$. Hence, if we reduce (1.2) to a local system **on** $\mathcal{M}_\eta^\sigma \cap U_2$, then we will have captured all of the dynamics of the problem. Remarks 4.1(d) is the first observation toward this reduced formulation. In fact, one can make use of the properties established in Remarks 4.1 to prove the following, even more general, result regarding the properties of the tangent vectors to \mathcal{M}_η^σ . Although we use other tools to connect the reduced problem (4.5) below with the full problem (1.2), this remark provides good intuition into the nature of these manifolds.

Remark 4.2. Given $(\tilde{\rho}, \eta) \in U$ it follows that $D_1\psi(\tilde{\rho}, \eta) \circ P_0 = id_{T_{\psi(\tilde{\rho}, \eta)}\mathcal{M}_\eta^\sigma}$, where $T_\rho\mathcal{M}_\eta^\sigma$ denotes the tangent space to the manifold \mathcal{M}_η^σ at the point ρ .

4.2. The Reduced Problem. Fix $\alpha \in (0, 1)$ and we denote the spaces

$$F_0 := h_0^\alpha(\mathbb{T}), \quad F_1 := h_0^{4+\alpha}(\mathbb{T}), \quad \text{and} \quad F_\mu := (F_0, F_1)_{\mu, \infty}^0, \quad \mu \in (0, 1),$$

so that $F_\mu = P_0 E_\mu$ for $\mu \in [0, 1]$. Define the operator

$$\mathcal{G}_\star(\tilde{\rho}, \eta) = \mathcal{G}_\star(\tilde{\rho}, \eta; r_\star) := P_0 G(\psi(\tilde{\rho}, \eta) + r_\star),$$

which is defined for all $(\tilde{\rho}, \eta) \in U \subset F_0 \times \mathbb{R}$ with $\tilde{\rho} \in U_0 \cap F_1$.

Now we consider the *reduced* problem for the zero-mean functions

$$(4.5) \quad \begin{cases} \tilde{\rho}_t(t, x) = \mathcal{G}_\star(\tilde{\rho}(t, x), \eta), & t > 0, x \in \mathbb{T}, \\ \tilde{\rho}(0, x) = \tilde{\rho}_0(x), & x \in \mathbb{T}, \end{cases}$$

where $\tilde{\rho}_0 := P_0 r_0 = P_0(r_0 - r_\star)$. One will note that we should insist on $\psi(\tilde{\rho}, \eta)(x) > -r_\star$ for all $x \in \mathbb{T}$ in order to guarantee that $G(\psi(\tilde{\rho}, \eta) + r_\star)$ is well-defined. However, we can assume, without loss of generality, that the neighborhood U is chosen small enough to ensure this property holds for all $(\tilde{\rho}, \eta) \in U$.

Remarks 4.3. Throughout most of the analysis that follows, we will treat the parameter η as a free parameter, although it has a very specific interpretation in relation to (2.1). If one is given initial data r_0 close to r_\star , then the parameter η is chosen so that

$$F_\star(\eta) = F_\star(r_0).$$

- a) Essentially, this parameter allows for the possibility that the volume enclosed by the surface $\Gamma(r_0)$ differs from that of the cylinder $\Gamma(r_\star)$, thereby allowing us to handle non-volume-preserving perturbations r_0 of the cylinder r_\star .
- b) From a more general viewpoint, one can see that the family $\{\mathcal{M}_\eta^\sigma \cap \psi(U) : \eta \in U_1\}$ forms a dimension 1 foliation of a neighborhood of the positive real axis $\mathbb{R}_+ \subset h^\sigma(\mathbb{T})$ and the parameter η separates the leaves of the foliation.

For $\mu \in (0, 1]$ and closed intervals $J \subseteq \mathbb{R}_+$ with $0 \in J$, define the spaces

$$\begin{aligned} \mathbb{E}_0(J) &:= BUC_{1-\mu}(J, E_0), \\ \mathbb{E}_1(J) &:= BUC_{1-\mu}^1(J, E_0) \cap BUC_{1-\mu}(J, E_1), \end{aligned}$$

and

$$\begin{aligned} \mathbb{F}_0(J) &:= BUC_{1-\mu}(J, F_0), \\ \mathbb{F}_1(J) &:= BUC_{1-\mu}^1(J, F_0) \cap BUC_{1-\mu}(J, F_1), \end{aligned}$$

within which we will discuss solutions to the shifted problem (4.1) and the reduced problem (4.5), respectively.

In order to connect these two problems, we will make use of the *lifting* map ψ , defined in the previous section. To ensure that ψ is well-defined on $\mathbb{F}_1(J)$, we must restrict our attention to functions which map into an appropriate neighborhood $U_0 \subset F_0$ of 0. In particular, we assume that U_0 is given so that

$$\psi(\cdot, \eta) : U_0 \subset F_0 \rightarrow E_0, \quad \eta \in U_1,$$

is in the regularity class C^ω and, without loss of generality, we assume that U_0 is given sufficiently small so that ψ and the derivative $D_1\psi$ are bounded on $U = U_0 \times U_1$. More precisely, U_0 is chosen sufficiently small so that there exists a constant $N > 0$ for which the inequalities

$$(4.6) \quad \|\psi(\tilde{\rho}, \eta)\|_{E_0} \leq N \quad \text{and} \quad \|D_1\psi(\tilde{\rho}, \eta)\|_{\mathcal{L}(F_0, E_0)} \leq N$$

hold for all $(\tilde{\rho}, \eta) \in U = U_0 \times U_1$.

Lemma 4.4. *Fix $\eta \in U_1$ and $J := [0, T]$ for $T > 0$. Then*

$$\psi(\cdot, \eta) : \mathbb{F}_1(J) \cap C(J, U_0) \longrightarrow \mathbb{E}_1(J), \quad \text{with} \quad \psi(\tilde{\rho}, \eta)(t) := \psi(\tilde{\rho}(t), \eta).$$

Moreover, if $\tilde{\rho}_0 \in F_\mu$ and $\tilde{\rho} = \tilde{\rho}(\cdot, \tilde{\rho}_0) \in \mathbb{F}_1(J) \cap C(J, U_0)$ is a solution to (4.5), for some $\mu \in [1/2, 1]$, then $\rho := \psi(\tilde{\rho}, \eta)$ is the unique solution on the interval J to (4.1), with initial data $\rho_0 := \psi(\tilde{\rho}_0, \eta) \in E_\mu$.

Proof. First notice that the embeddings

$$(4.7) \quad \mathbb{F}_1(J) \hookrightarrow BUC(J, F_\mu) \hookrightarrow BUC(J, F_0), \quad \mu \in [1/2, 1],$$

follow from [2, Theorem III.2.3.3] and the continuous embedding of little-Hölder spaces, respectively.

To see that the mapping property for $\psi(\cdot, \eta)$ holds, let $\tilde{\rho} \in \mathbb{F}_1(J) \cap C(J, U_0)$. Uniform continuity and differentiability of the function $\psi(\tilde{\rho}(\cdot), \eta)$ follows from the regularity of ψ and $\tilde{\rho}$, and compactness of the interval J . Hence we focus on demonstrating that $\psi(\tilde{\rho}(\cdot), \eta)$ satisfies the boundedness conditions for $\mathbb{E}_1(J)$. In the case $\mu \in [1/2, 1)$, it follows from Remarks 4.1(f) and (4.6) that, for $t \in J$,

$$(4.8) \quad \begin{aligned} t^{1-\mu} \|\psi(\tilde{\rho}(t), \eta)\|_{E_1} &\leq t^{1-\mu} \|\tilde{\rho}(t)\|_{F_1} + \frac{t^{1-\mu}}{2\pi} \int_{\mathbb{T}} |\psi(\tilde{\rho}(t), \eta)(x)| dx \\ &\leq \|\tilde{\rho}\|_{\mathbb{F}_1(J)} + t^{1-\mu} \|\psi(\tilde{\rho}(t), \eta)\|_{C(\mathbb{T})} \\ &\leq \|\tilde{\rho}\|_{\mathbb{F}_1(J)} + T^{1-\mu} N, \end{aligned}$$

$$\text{and} \quad \lim_{t \rightarrow 0} t^{1-\mu} \|\psi(\tilde{\rho}(t), \eta)\|_{E_1} = 0.$$

From (4.8) we conclude that $\psi(\tilde{\rho}, \eta) \in BUC_{1-\mu}(J, E_1)$. Meanwhile, looking at the time derivative of $\psi(\tilde{\rho}, \eta)$, we note that $\partial_t \psi(\tilde{\rho}(t), \eta) = D_1\psi(\tilde{\rho}(t), \eta) \partial_t \tilde{\rho}(t)$ and so we again make use of (4.6) to see that

$$\begin{aligned} t^{1-\mu} \|\partial_t \psi(\tilde{\rho}(t), \eta)\|_{E_0} &\leq \|D_1\psi(\tilde{\rho}(t), \eta)\|_{\mathcal{L}(F_0, E_0)} t^{1-\mu} \|\partial_t \tilde{\rho}(t)\|_{F_0} \\ &\leq N \|\tilde{\rho}\|_{\mathbb{F}_1(J)} < \infty, \end{aligned}$$

$$\text{and} \quad \lim_{t \rightarrow 0} t^{1-\mu} \|\partial_t \psi(\tilde{\rho}(t), \eta)\|_{E_0} = 0.$$

Hence, making use of the embedding $E_1 \hookrightarrow E_0$, we see that $\psi(\tilde{\rho}, \eta) \in \mathbb{E}_1(J)$, as desired. Meanwhile, when $\mu = 1$ we again get continuity and differentiability from the regularity of the mappings $\tilde{\rho}$ and ψ .

To see that the second part of the lemma holds, observe by (4.7) that $\rho_0 := \psi(\tilde{\rho}_0, \eta) \in E_\mu \cap U_\star$. Hence, by Theorem 2.2, there exists a unique maximal solution

$$r(\cdot, \rho_0) \in C_{1-\mu}^1(J(\rho_0), E_0) \cap C_{1-\mu}(J(\rho_0), E_1)$$

to (4.1) on some maximal interval of existence $J(\rho_0) = [0, t^+(\rho_0))$. Now, define $\rho(\cdot) := \psi(\tilde{\rho}(\cdot), \eta)$ as indicated and it suffices to show that $\rho_t(t) = G_\star(\rho(t))$ for $t \in \dot{J} := (0, T]$, since this will imply that $\rho(t) = r(t, \rho_0)$ by uniqueness and maximality of the solution $r(\cdot, \rho_0)$. Proceeding, let $t \in \dot{J}$ and consider the auxiliary problem

$$\begin{cases} \dot{\gamma}(\tau) = G_\star(\gamma(\tau)), & \text{for } \tau \in [0, \varepsilon], \\ \gamma(0) = \rho(t), \end{cases}$$

which has a unique solution $\gamma \in C^1([0, \varepsilon], E_0) \cap C([0, \varepsilon], E_1)$ by Theorem 2.2, provided we choose $\varepsilon > 0$ sufficiently small for the particular value $\rho(t) \in E_1$. Notice, by the regularity of γ , we have

$$\dot{\gamma}(0) = G_\star(\gamma(0)) = G_\star(\rho(t)).$$

Further, note that $\rho(t) \in \mathcal{M}_\eta^{4+\alpha}$, from which we conclude that $\gamma(\tau) \in \mathcal{M}_\eta^{4+\alpha}$ and by Remarks 4.1 we have the representation $\gamma(\tau) = \psi(P_0\gamma(\tau), \eta)$, $\tau \in [0, \varepsilon]$. Finally, we see that

$$\begin{aligned} G_\star(\rho(t)) &= \dot{\gamma}(0) = \partial_\tau (\psi(P_0\gamma(\tau), \eta)) \Big|_{\tau=0} = D_1\psi(P_0\gamma(0), \eta)P_0\dot{\gamma}(0) \\ (4.9) \quad &= D_1\psi(P_0\rho(t), \eta)P_0G_\star(\rho(t)) = D_1\psi(\tilde{\rho}(t), \eta)\mathcal{G}_\star(\tilde{\rho}(t), \eta) \\ &= \partial_t (\psi(\tilde{\rho}(t), \eta)) = \rho_t(t), \end{aligned}$$

which concludes the proof. \square

We also get the following results which further illuminate the relationship between the mappings G_\star and \mathcal{G}_\star , and explicitly connects the equilibria of the two problems (4.1) and (4.5).

Lemma 4.5. *For any $\rho \in \mathcal{M}_\eta^{4+\alpha} \cap U_2$, it follows that*

$$(4.10) \quad G_\star(\rho) = D_1\psi(P_0\rho, \eta)P_0G_\star(\rho),$$

and

$$(4.11) \quad DG_\star(\rho)h = D_1^2\psi(P_0\rho, \eta)[P_0h, P_0G_\star(\rho)] + D_1\psi(P_0\rho, \eta)P_0DG_\star(\rho)h,$$

for $h \in E_1$.

Proof. The first claim was justified in the proof of Lemma 4.4 above and is expressed in (4.9). Meanwhile, the second claim follows immediately by differentiation. \square

Proposition 4.6. *If $(\tilde{\rho}, \eta) \in U$, then $(\tilde{\rho}, \eta)$ is an equilibrium of (4.5) if and only if $\psi(\tilde{\rho}, \eta)$ is an equilibrium of (4.1), i.e.*

$$\mathcal{G}_\star(\tilde{\rho}, \eta) = 0 \iff G_\star(\psi(\tilde{\rho}, \eta)) = 0.$$

Moreover, if $\mathcal{G}_\star(\tilde{\rho}, \eta) = 0$, then it follows that

$$(4.12) \quad DG_\star(\psi(\tilde{\rho}, \eta))h = D_1\psi(\tilde{\rho}, \eta)P_0DG_\star(\psi(\tilde{\rho}, \eta))h, \quad h \in E_1,$$

and

$$(4.13) \quad DG_\star(\psi(\tilde{\rho}, \eta))D_1\psi(\tilde{\rho}, \eta)\tilde{h} = D_1\psi(\tilde{\rho}, \eta)D_1\mathcal{G}_\star(\tilde{\rho}, \eta)\tilde{h}, \quad \tilde{h} \in F_1.$$

Proof. The first claim follows from the definition of \mathcal{G}_\star and (4.10), while (4.12) is a consequence of (4.11) and (4.13) follows from (4.11) and the chain rule:

$$\begin{aligned} DG_\star(\psi(\tilde{\rho}, \eta)) D_1 \psi(\tilde{\rho}, \eta) \tilde{h} &= D_1 \psi(\tilde{\rho}, \eta) P_0 DG_\star(\psi(\tilde{\rho}, \eta)) D_1 \psi(\tilde{\rho}, \eta) \tilde{h} \\ &= D_1 \psi(\tilde{\rho}, \eta) D_1 \mathcal{G}_\star(\tilde{\rho}, \eta) \tilde{h}. \end{aligned} \quad \square$$

4.3. Mapping Properties of $D_1 \mathcal{G}_\star(0, \eta)$. Notice that the points $(0, \eta) \in U$ are equilibria of (4.5), and they correspond to the cylinders $\Gamma(r_\star + \eta)$. We are interested in the spectral properties of the linearization of \mathcal{G}_\star about these equilibria. In particular, we compute the Fréchet derivative

$$D_1 \mathcal{G}_\star(0, \eta) h = P_0 DG_\star(\psi(0, \eta)) D_1 \psi(0, \eta) h = P_0 DG_\star(\eta) D_1 \psi(0, \eta) h,$$

for $h \in F_1$. Hence, by Remarks 4.1(d) we derive the formula

$$(4.14) \quad D_1 \mathcal{G}_\star(0, \eta) = P_0 DG_\star(\eta)|_{F_1} = DG_\star(\eta)|_{F_1},$$

where the last equality is verified by application of the divergence theorem to the linearization

$$(4.15) \quad DG_\star(\eta) = -\partial_x^2 \left(\frac{1}{(r_\star + \eta)^2} + \partial_x^2 \right).$$

Utilizing the Fourier series representation of functions in $h^\sigma(\mathbb{T})$, c.f. [29, Propositions 1.2 and 1.3], we find the eigenvalues of this linearized operator. In particular, for $h \in E_1$,

$$\begin{aligned} (\lambda - DG_\star(\eta)) h &= \left(\lambda + \partial_x^2 \left(\frac{1}{(r_\star + \eta)^2} + \partial_x^2 \right) \right) \sum_{k \in \mathbb{Z}} \hat{h}(k) e_k \\ &= \sum_{k \in \mathbb{Z}} \left(\lambda - k^2 \left(\frac{1}{(r_\star + \eta)^2} - k^2 \right) \right) \hat{h}(k) e_k \\ (4.16) \quad \implies \quad \sigma_p(DG_\star(\eta)) &= \left\{ k^2 \left(\frac{1}{(r_\star + \eta)^2} - k^2 \right) : k \in \mathbb{Z} \right\}. \end{aligned}$$

Noting that the embedding $E_1 \hookrightarrow E_0$ is compact, it follows that the resolvent $R(\lambda) := (\lambda - DG_\star(\eta))^{-1}$ is a compact operator, λ in the resolvent set $\rho(DG_\star(\eta))$. It follows from classic theory of linear operators that the spectrum $\sigma(DG_\star(\eta))$ consists entirely of isolated eigenvalues of finite multiplicity, see Kato [25, Theorem III.6.29] for instance. Hence, $\sigma_p(DG_\star(\eta)) = \sigma(DG_\star(\eta))$.

Remark 4.7. If $r_\star + \eta > 1$, then $\sigma(DG_\star(\eta)) \subset (-\infty, 0]$, however the spectrum will always contain 0. The presence of this 0 eigenvalue can be seen as a consequence of the fact that the equilibria $r_\star + \eta$ are not isolated in the space E_1 . Hence, by passing to the operator \mathcal{G}_\star , which acts on an open subset of the zero-mean functions F_1 , we eliminate the nontrivial equilibria (since the only constant function in F_1 is the zero function) and thereby eliminate the zero eigenvalue. In particular, one easily computes that

$$(4.17) \quad \sigma(D_1 \mathcal{G}_\star(0, \eta)) = \left\{ k^2 \left(\frac{1}{(r_\star + \eta)^2} - k^2 \right) : k \in \mathbb{Z} \setminus \{0\} \right\}, \quad \eta \in U_1.$$

Before we return to the problem (1.2), we state the following maximal regularity result for the linearization $D_1 \mathcal{G}_\star(0, \eta)$. For this result, we define the exponentially

weighted maximal regularity spaces

$$\mathbb{F}_j(\mathbb{R}_+, \omega) := \left\{ f : (0, \infty) \rightarrow F_0 \mid [t \mapsto e^{\omega t} f(t)] \in \mathbb{F}_j(\mathbb{R}_+) \right\}, \quad \omega \in \mathbb{R}, j = 0, 1,$$

which are Banach spaces when equipped with the norms $\|u\|_{\mathbb{F}_j(\mathbb{R}_+, \omega)} := \|e^{\omega t} u\|_{\mathbb{F}_j(\mathbb{R}_+)}$.

Theorem 4.8. *Suppose $r_\star > 1$ and $\mu \in (0, 1]$. There exist nonzero positive constants $\delta = \delta(r_\star)$ and $\omega = \omega(r_\star, \delta)$ such that*

$$\left(\mathbb{F}_0(\mathbb{R}_+, \omega), \mathbb{F}_1(\mathbb{R}_+, \omega) \right)$$

is a pair of maximal regularity for $-D_1 \mathcal{G}_\star(0, \eta)$, given any $\eta \in (-\delta, \delta)$. I.e. the property

$$(\partial_t - D_1 \mathcal{G}_\star(0, \eta), \gamma) \in \mathcal{L}_{isom} \left(\mathbb{F}_1(\mathbb{R}_+, \omega), \mathbb{F}_0(\mathbb{R}_+, \omega) \times h_0^{4\mu+\alpha}(\mathbb{T}) \right),$$

holds uniformly for $\eta \in (-\delta, \delta)$.

Proof. Fix $\delta > 0$ so that $(-\delta, \delta) \subset U_1 \cap (1 - r_\star, \infty)$. Following the notation and definitions of [29], it is clear from the representation (4.15) that $-DG_\star(\eta)$ is a uniformly elliptic operator from which we see, by [29, Theorem 4.4], that $DG_\star(\eta)$ generates an analytic semigroup on $h^\alpha(\mathbb{T}, \mathbb{C})$ with domain $h^{4+\alpha}(\mathbb{T}, \mathbb{C})$. Since $h_0^\alpha(\mathbb{T}, \mathbb{C})$ inherits the topology of $h^\alpha(\mathbb{T}, \mathbb{C})$ and the projection P_0 commutes with $DG_\star(\eta)$, the analogous resolvent estimates hold for $D_1 \mathcal{G}_\star(0, \eta)$ and so we conclude that $D_1 \mathcal{G}_\star(0, \eta)$ generates an analytic semigroup on $h_0^\alpha(\mathbb{T}, \mathbb{C})$ with domain $h_0^{4+\alpha}(\mathbb{T}, \mathbb{C})$. Moreover, from (4.17) we see that $\text{type}(D_1 \mathcal{G}_\star(0, \eta)) < 0$ for all $\eta \in (-\delta, \delta)$, where $\text{type}(B)$ denotes the spectral type of the semigroup generator B . In particular, it follows that

$$\text{type}(D_1 \mathcal{G}_\star(0, \eta)) < \frac{1 - (r_\star - \delta)^2}{(r_\star - \delta)^2} < 0, \quad \eta \in (-\delta, \delta).$$

Now, choose $\omega \in \left(0, \frac{(r_\star - \delta)^2 - 1}{(r_\star - \delta)^2}\right)$ and the remainder of the result follows from [2, Theorem III.3.4.1 and Remarks 3.4.2(b)] and the restriction of maximal regularity to the subspaces $h_0^\sigma(\mathbb{T})$. Notice, the characterization $\gamma \mathbb{F}_1(\omega) = h_0^{4\mu+\alpha}(\mathbb{T})$ follows from stability of little-Hölder spaces under continuous interpolation. \square

4.4. Exponential Stability of Cylinders With Radius $r_\star > 1$. Our main result regarding stability of cylinders in the axisymmetric surface diffusion flow (1.2) establishes exponential asymptotic stability of the family of cylinders under perturbations which maintain the prescribed periodic boundary conditions and symmetry about the same axis of rotation (which we are taking to be the x-axis in our setting). One feature of our result that we point out is the fact that it establishes a form of stability which allows for perturbations which are not volume-preserving. In particular, we refer to *asymptotic stability* of the cylinder $\Gamma(r_\star)$ by which we mean that small perturbations of $\Gamma(r_\star)$ will have global solutions to (1.2) which converge to a cylinder $\Gamma(r_\star + \eta)$, where $r_\star \neq r_\star + \eta$ in general.

Theorem 4.9 (Exponential Stability). *Fix $\alpha \in (0, 1)$, $\mu \in [1/2, 1]$, so that $4\mu + \alpha \notin \mathbb{Z}$, and $r_\star > 1$. There exist nonzero positive constants $\varepsilon = \varepsilon(r_\star)$, $\delta = \delta(r_\star)$ and $\omega = \omega(r_\star, \delta)$, such that problem (2.1) with initial data $r_0 \in \mathbb{B}_{h^{4\mu+\alpha}}(r_\star, \varepsilon)$ has a unique global solution*

$$r(\cdot, r_0) \in C_{1-\mu}^1(\mathbb{R}_+, h^\alpha(\mathbb{T})) \cap C_{1-\mu}(\mathbb{R}_+, h^{4+\alpha}(\mathbb{T})),$$

and there exists $\eta = \eta(r_0) \in (-\delta, \delta)$ and $M = M(\alpha) > 0$ for which the bound

$$t^{1-\mu} \|r(t, r_0) - (r_\star + \eta)\|_{h^{4+\alpha}} + \|r(t, r_0) - (r_\star + \eta)\|_{h^{4\mu+\alpha}} \leq e^{-\omega t} M \|r_0 - r_\star\|_{h^{4\mu+\alpha}}$$

holds uniformly for $t > 0$.

Proof. (i) Let $\delta, \omega > 0$ be the constants given by Theorem 4.8 and consider the operator

$$\mathcal{K}(\tilde{\rho}, \tilde{\rho}_0, \eta) := \left(\partial_t \tilde{\rho} - \mathcal{G}_\star(\tilde{\rho}, \eta), \gamma \tilde{\rho} - \tilde{\rho}_0 \right),$$

acting on $\mathbb{U} := \left(\mathbb{F}_1(\mathbb{R}_+, \omega) \cap C(\mathbb{R}_+, U_0) \right) \times \left(U_0 \cap F_\mu \right) \times U_1$ which is open in the Banach space $\mathbb{F}_1(\mathbb{R}_+, \omega) \times F_\mu \times \mathbb{R}$.

Considering the space into which \mathcal{K} maps, first notice that

$$\gamma : \mathbb{F}_1(\mathbb{R}_+, \omega) \rightarrow (F_0, F_1)_{\mu, \infty}^0$$

follows from [9, Lemma 2.2(a)]. Hence, $\gamma \tilde{\rho} \in F_\mu$ and ∂_t maps $\mathbb{F}_1(\mathbb{R}_+, \omega)$ into $\mathbb{F}_0(\mathbb{R}_+, \omega)$ by definition of the spaces $BUC_{1-\mu}^1(J, E)$. To see that $\mathcal{G}_\star(\cdot, \eta)$ maps \mathbb{U} into $\mathbb{F}_0(\mathbb{R}_+, \omega)$, choose $\tilde{\rho} \in \mathbb{U}$ and notice that $\tilde{\rho}(t) \in U_0 \cap h_0^{2+\alpha}(\mathbb{T})$, for $t > 0$, from the embeddings (4.7). Utilizing the explicit quasilinear representation of the operator G , as given by (2.2)–(2.3), whereby

$$\mathcal{G}_\star(\tilde{\rho}(t), \eta) = P_0 \left(-\mathcal{A}(\psi(\tilde{\rho}(t), \eta) + r_\star)(\psi(\tilde{\rho}(t), \eta) + r_\star) + f(\psi(\tilde{\rho}(t), \eta) + r_\star) \right),$$

one will easily conclude the desired mapping property for the operator \mathcal{G}_\star . For instance, we have seen that $\mathcal{A}(\rho)\rho = b_1(\rho)\partial_x^4 \rho + b_2(\rho)\partial_x^3 \rho$, where the functions b_i only depend on ρ, ρ_x and ρ_{xx} , $i = 1, 2$. Hence, it follows that

$$\begin{aligned} & e^{\omega t} t^{1-\mu} \left\| \mathcal{A}(\psi(\tilde{\rho}(t), \eta) + r_\star)(\psi(\tilde{\rho}(t), \eta) + r_\star) \right\|_{E_0} \\ & \leq e^{\omega t} t^{1-\mu} \left\| \partial_x^4 \psi(\tilde{\rho}(t), \eta) \right\|_{E_0} \left\| b_1(\psi(\tilde{\rho}(t), \eta) + r_\star) \right\|_{E_0} \\ & \quad + e^{\omega t} t^{1-\mu} \left\| \partial_x^3 \psi(\tilde{\rho}(t), \eta) \right\|_{E_0} \left\| b_2(\psi(\tilde{\rho}(t), \eta) + r_\star) \right\|_{E_0}, \end{aligned}$$

for $t > 0$. From here, we take advantage of the boundedness of $\psi(\tilde{\rho}(t), \eta)$ in the topology of $F_{1/2}$, in conjunction with the explicit formulas for b_i , in order to bound the terms $\|b_i(\psi(\tilde{\rho}(t), \eta) + r_\star)\|_{E_0}$, uniformly in t . Meanwhile, the representation given by Remarks 4.1(d) and the fact that $\tilde{\rho} \in \mathbb{F}_1(\mathbb{R}_+, \omega)$ yield the bounds

$$e^{\omega t} t^{1-\mu} \left\| \partial_x^k \psi(\tilde{\rho}(t), \eta) \right\|_{E_0} = e^{\omega t} t^{1-\mu} \left\| \partial_x^k \tilde{\rho}(t) \right\|_{F_0} \leq \|e^{\omega t} \tilde{\rho}\|_{\mathbb{F}_1(\mathbb{R}_+)}, \quad k = 1, \dots, 4.$$

Analogous methods work for the remaining terms of the function $G_\star(\psi(\tilde{\rho}(t), \eta))$, since we can always isolate an element of the form $\partial_x^k \psi(\tilde{\rho}(t), \eta)$, and bound the remaining elements using boundedness in $F_{1/2}$. We conclude the result by noting that the linear projection P_0 adds no complexity to acquiring the necessary bounds.

With the establishment of the spaces into which the operator \mathcal{K} maps, we move on with our analysis. Regarding the regularity of \mathcal{K} , it can be shown that \mathcal{G}_\star is C^ω via substitution operators and the derivative ∂_t and the trace operator γ are linear. Hence, it follows that

$$\mathcal{K} \in C^\omega \left(\mathbb{U}, \mathbb{F}_0(\mathbb{R}_+, \omega) \times F_\mu \right).$$

Meanwhile, notice that $\mathcal{K}(0, 0, 0) = (0, 0)$ and

$$D_1 \mathcal{K}(0, 0, 0) = \left(\partial_t - D_1 \mathcal{G}_\star(0, 0), \gamma \right) \in \mathcal{L}_{isom} \left(\mathbb{F}_1(\mathbb{R}_+, \omega), \mathbb{F}_0(\mathbb{R}_+, \omega) \times F_\mu \right),$$

by Theorem 4.8. Hence, we conclude from the implicit function theorem that there exists an open neighborhood $0 \in \tilde{U} \subset F_\mu \times \mathbb{R}$ and a C^ω mapping $\kappa : \tilde{U} \rightarrow \mathbb{F}_1(\mathbb{R}_+, \omega)$ such that

$$\mathcal{K}(\kappa(\tilde{\rho}_0, \eta), \tilde{\rho}_0, \eta) = (0, 0) \quad \text{for all } (\tilde{\rho}_0, \eta) \in \tilde{U}.$$

In particular, $\kappa(\tilde{\rho}_0, \eta)$ is a global solution to (4.5) with parameter η and initial data $\tilde{\rho}_0 \in F_\mu$, where we assume, without loss of generality, that $\tilde{U} \subseteq U$.

(ii) Choose $\varepsilon > 0$ so that for every $r_0 \in \mathbb{B}_{F_\mu}(r_\star, \varepsilon)$, there exists $\eta \in (-r_\star, \infty)$ for which

$$(P_0 r_0, \eta) \in \tilde{U} \quad \text{and} \quad F_\star(r_0 - r_\star; r_\star) = F_\star(\eta; r_\star).$$

The existence of such a constant ε is guaranteed by the continuity of P_0 and F_\star , injectivity of $F_\star(\eta; r_\star)$ for $\eta \in (-r_\star, \infty)$ and the fact that $P_0 r_\star = 0$.

Let $r_0 \in \mathbb{B}_{F_\mu}(r_\star, \varepsilon)$ and fix $\eta = \eta(r_0)$ as mentioned so that $F_\star(r_0 - r_\star; r_\star) = F_\star(\eta; r_\star)$. Define the function

$$(4.18) \quad r := \psi(\kappa(P_0 r_0, \eta), \eta) + r_\star,$$

where $\psi(\kappa(P_0 r_0, \eta), \eta)(t) := \psi(\kappa(P_0 r_0, \eta)(t), \eta)$, and we will demonstrate that r satisfies the desired properties claimed in the theorem.

To see that r is the unique global solution to (2.1) with initial data r_0 , first fix $T > 0$ and consider the interval $J := [0, T]$. By the choice of $\varepsilon > 0$ we know that $(P_0 r_0, \eta) \in \tilde{U}$ and so it follows from part (i) above that $\kappa(P_0 r_0, \eta) \in \mathbb{F}_1(\mathbb{R}_+, \omega)$. From this we see that $\kappa(P_0 r_0, \eta) \in \mathbb{F}_1(J)$ is a solution to (4.5) with initial data $P_0 r_0 \in F_\mu$. Thus it follows, by Lemma 4.4, that $r \in \mathbb{E}_1(J)$ is the solution on J to the problem (2.1) with initial data

$$\psi(P_0 r_0, \eta) + r_\star = \psi(P_0(r_0 - r_\star), \eta) + r_\star = r_0,$$

where we use Remarks 4.1(b) and the fact that $r_0 - r_\star \in \mathcal{M}_\eta^{4\mu+\alpha}$. The claim now follows by the fact that $T > 0$ was arbitrary and by definition of the Fréchet spaces $C_{1-\mu}(\mathbb{R}_+, E)$.

Now, to see that r satisfies the exponential bounds in the second part of the claim, first notice that $\kappa(0, \eta) \equiv 0$ for $\eta \in U_1$. Then, by definition of r , Remarks 4.1, and application of the mean value theorem, we see that the expression

$$\begin{aligned} r(t) - (r_\star + \eta) &= \psi(\kappa(P_0 r_0, \eta)(t), \eta) - \eta = \psi(\kappa(P_0 r_0, \eta)(t), \eta) - \psi(\kappa(0, \eta)(t), \eta) \\ &= \left(P_0 + (1 - P_0) \right) \left(\psi(\kappa(P_0 r_0, \eta)(t), \eta) - \psi(\kappa(0, \eta)(t), \eta) \right) \\ &= \kappa(P_0 r_0, \eta)(t) + \frac{1}{2\pi} \int_{\mathbb{T}} \left(\psi(\kappa(P_0 r_0, \eta)(t, x), \eta) - \psi(\kappa(0, \eta)(t, x), \eta) \right) dx \\ &= \kappa(P_0 r_0, \eta)(t) + \frac{1}{2\pi} \int_{\mathbb{T}} \int_0^1 D_1 \psi(\tau \kappa(P_0 r_0, \eta)(t), \eta) \kappa(P_0 r_0, \eta)(t, x) d\tau dx, \end{aligned}$$

holds for all $t > 0$. Notice that

$$e^{\omega t} t^{1-\mu} \|\kappa(P_0 r_0, \eta)(t)\|_{F_1} \leq \|\kappa(P_0 r_0, \eta)\|_{\mathbb{F}_1(\mathbb{R}_+, \omega)}$$

and

$$\sup_{t \in \mathbb{R}_+} \|e^{\omega t} \kappa(P_0 r_0, \eta)(t)\|_{F_\mu}$$

are finite quantities by the fact that $\kappa(P_0 r_0, \eta) \in \mathbb{F}_1(\mathbb{R}_+, \omega)$ and the embedding (4.7). We note that the reference for (4.7) does not explicitly include the unbounded interval $J = \mathbb{R}_+$, however the methods of the proof extend to this unbounded case with little trouble. Meanwhile, the remaining term in $r(t) - (r_\star + \eta)$ above is scalar-valued, so we bound $D_1 \psi(\tau \kappa(P_0 r_0, \eta)(t), \eta) \kappa(P_0 r_0, \eta)(t)$ in the $C(\mathbb{T})$ -topology, which are then bounded in the $h^\sigma(\mathbb{T})$ -topology, for any $\sigma \in \mathbb{R}_+ \setminus \mathbb{Z}$. In particular, observe that, by (4.6),

$$\sup_{\tilde{\rho} \in U_0} \|D_1 \psi(\tilde{\rho}, \eta) \kappa(P_0 r_0, \eta)(t)\|_{h^\alpha} \leq N \|\kappa(P_0 r_0, \eta)(t)\|_{h_0^\alpha}, \quad t > 0,$$

and we conclude that the bounds

$$(4.19) \quad e^{\omega t} t^{1-\mu} \|r(t) - (r_\star + \eta)\|_{E_1} \leq (1 + c_1 N) \|\kappa(P_0 r_0, \eta)\|_{\mathbb{F}_1(\mathbb{R}_+, \omega)}$$

and

$$(4.20) \quad e^{\omega t} \|r(t) - (r_\star + \eta)\|_{E_\mu} \leq (c_2 + c_3 N) \|\kappa(P_0 r_0, \eta)\|_{\mathbb{F}_1(\mathbb{R}_+, \omega)},$$

hold uniformly for $t > 0$. Here the constant c_1 comes from the embedding $F_1 \hookrightarrow F_0$, and the constants c_2 and c_3 come from the embeddings (4.7). Finally, by the regularity of κ , we may assume that \tilde{U} was chosen sufficiently small to ensure that $D_1 \kappa$ is uniformly bounded from \tilde{U} into $\mathbb{F}_1(\mathbb{R}_+, \omega)$. Recalling that $\kappa(0, \eta) = 0$, it follows that

$$(4.21) \quad \begin{aligned} \|\kappa(P_0 r_0, \eta)\|_{\mathbb{F}_1(\mathbb{R}_+, \omega)} &\leq \int_0^1 \|D_1 \kappa(\tau P_0 r_0, \eta) P_0 r_0\|_{\mathbb{F}_1(\mathbb{R}_+, \omega)} d\tau \\ &\leq \tilde{M} \|P_0 r_0\|_{F_\mu} \leq M \|r_0 - r_\star\|_{E_\mu}, \end{aligned}$$

where $M := \|P_0\| \sup_{(\tilde{\rho}, \eta) \in \tilde{U}} \|D_1 \kappa(\tilde{\rho}, \eta)\|_{\mathcal{L}(F_\mu, \mathbb{F}_1(\mathbb{R}_+, \omega))}$. The claim now follows from (4.21) and the inequalities (4.19)–(4.20). \square

5. INSTABILITY OF CYLINDERS WITH RADIUS $0 < r_\star < 1$

In this section we turn our attention to the stability of cylinders with small radius. Again, taking advantage of the reduced problem (4.5) and the connection we have established between it and the original problem (1.2), we proceed with the following result regarding instability of cylinders with radius $0 < r_\star < 1$, in the setting of F_μ . Because of the nature of the instability result, differences in volume between the initial data r_0 and the cylinder r_\star are not a factor in the following argument. In light of this, we will assume that the parameter η , associated with the reduced problem (4.5), is simply taken to be zero for this proof.

Theorem 5.1. *Let $r_\star \in (0, 1)$ and $\mu \in [1/2, 1]$ be fixed with $4\mu + \alpha \notin \mathbb{Z}$. Then the equilibrium 0 of (4.5) is unstable in the topology of $h_0^{4\mu+\alpha}(\mathbb{T})$ for initial values in $h_0^{4\mu+\alpha}(\mathbb{T})$.*

Proof. (i) Let $r_\star \in (0, 1)$ be fixed, and let $L := D_1 \mathcal{G}_\star(0, 0)$ be the linearization of \mathcal{G}_\star at $\tilde{\rho} = 0$. We can restate the evolution equation (4.5) in the following equivalent form

$$(5.1) \quad \begin{cases} \tilde{\rho}_t - L\tilde{\rho} = g(\tilde{\rho}), & t > 0 \\ \tilde{\rho}(0) = \tilde{\rho}_0, \end{cases}$$

where $g(\tilde{\rho}) := \mathcal{G}_\star(\tilde{\rho}, 0) - L\tilde{\rho}$. Using the quasilinear structure of $[\tilde{\rho} \mapsto \mathcal{G}_\star(\tilde{\rho}, 0)]$ it is not difficult to see that for every $\beta > 0$ there exists a number $\varepsilon_0 = \varepsilon_0(\beta) > 0$ such that

$$(5.2) \quad \|g(\tilde{\rho})\|_{F_0} \leq \beta \|\tilde{\rho}\|_{F_1}, \quad \tilde{\rho} \in \mathbb{B}_{F_\mu}(0, \varepsilon_0) \cap F_1,$$

where we will be assuming throughout that $\tilde{\rho} \in U_0$, to guarantee that $\mathcal{G}_\star(\tilde{\rho}, 0)$, and subsequently $g(\tilde{\rho})$, is defined. It follows from (4.17) that

$$\sigma(L) \cap [\operatorname{Re} z > 0] \neq \emptyset,$$

and we may choose numbers $\omega, \gamma > 0$ such that

$$[\omega - \gamma \leq \operatorname{Re} z \leq \omega + \gamma] \cap \sigma(L) = \emptyset \quad \text{and} \quad \sigma_+ := [\operatorname{Re} z > \omega + \gamma] \cap \sigma(L) \neq \emptyset,$$

i.e. the strip $[\omega - \gamma \leq \operatorname{Re} z \leq \omega + \gamma]$ does not intersect $\sigma(L)$ and there is at least one point of $\sigma(L)$ to the right of the line $[\operatorname{Re} z = \omega + \gamma]$.

We define P_+ to be the spectral projection, in F_0 , with respect to the spectral set σ_+ , and let $P_- := 1 - P_+$. Then $P_+(F_0)$ is finite dimensional and the topological decomposition

$$F_0 = P_+(F_0) \oplus P_-(F_0)$$

reduces L , so that $L = L_+ \oplus L_-$, where L_\pm is the part of L in $P_\pm(F_0)$, respectively, with the domains $D(L_\pm) = P_\pm(F_1)$. Moreover, P_\pm decomposes F_1 by the embedding $F_1 \hookrightarrow F_0$, and, without loss of generality, we can take the norm on F_1 so that

$$\|v\|_{F_1} = \|P_+v\|_{F_1} + \|P_-v\|_{F_1}.$$

We note that

$$\sigma(L_-) \subset [\operatorname{Re} z < \omega - \gamma], \quad \sigma(L_+) = \sigma^+ \subset [\operatorname{Re} z > \omega + \gamma].$$

This implies that there is a constant $M_0 \geq 1$ such that

$$(5.3) \quad \begin{aligned} \|e^{L_-t}P_-\|_{\mathcal{L}(F_0)} &\leq M_0 e^{(\omega-\gamma)t}, \\ \|e^{-L_+t}P_+\|_{\mathcal{L}(F_0)} &\leq M_0 e^{-(\omega+\gamma)t}, \quad t \geq 0, \end{aligned}$$

where $\{e^{L_-t} : t \geq 0\}$ is the analytic semigroup in $P_-(F_0)$ generated by L_- and $\{e^{L_+t} : t \in \mathbb{R}\}$ is the group in $P_+(F_0)$ generated by the bounded operator L_+ .

From (4.14)–(4.15) and [29, Theorem 5.2] one sees that $(\mathbb{F}_0(J), \mathbb{F}_1(J))$ is a pair of maximal regularity for $-L$ and it is easy to see that $-L_-$ inherits the property of maximal regularity. In particular, the pair $(P_-(\mathbb{F}_0(J)), P_-(\mathbb{F}_1(J)))$ is a pair of maximal regularity for $-L_-$. In fact, since $\operatorname{type}(-\omega + L_-) < -\gamma < 0$ we see that $(P_-(\mathbb{F}_0(\mathbb{R}_+)), P_-(\mathbb{F}_1(\mathbb{R}_+)))$ is a pair of maximal regularity for $(\omega - L_-)$. This, in turn, implies the a priori estimate

$$(5.4) \quad \|e^{-\omega t}w\|_{\mathbb{F}_1(J_T)} \leq M_1 \left(\|w_0\|_{F_\mu} + \|e^{-\omega t}f\|_{\mathbb{F}_0(J_T)} \right)$$

for $J_T := [0, T]$, any $T \in (0, \infty)$ (or $J_T = \mathbb{R}_+$ for $T = \infty$), with a universal constant $M_1 > 0$, where w is a solution of the linear Cauchy problem

$$\begin{cases} \dot{w} - L_-w = f, \\ w(0) = w_0, \end{cases}$$

with $(f, w_0) \in (C((0, T), P_-F_0), P_-U_0)$.

(ii) By way of contradiction, suppose that the equilibrium 0 is stable for (4.5). Then for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that (5.1) admits for each $\tilde{\rho}_0 \in \mathbb{B}_{F_\mu}(0, \delta)$ a global solution

$$\tilde{\rho} = \tilde{\rho}(\cdot, \tilde{\rho}_0) \in C_{1-\mu}^1(\mathbb{R}_+, F_0) \cap C_{1-\mu}(\mathbb{R}_+, F_1) \cap C(\mathbb{R}_+, U_0),$$

which satisfies

$$(5.5) \quad \|\tilde{\rho}(t)\|_{F_\mu} < \varepsilon, \quad t \geq 0.$$

We can assume without loss of generality that β and ε are chosen such that

$$(5.6) \quad 2C_0(M_0 + M_1\gamma)\beta \leq \gamma \quad \text{and} \quad \varepsilon \leq \varepsilon_0(\beta),$$

where $C_0 := \max\{\|P_-\|_{\mathcal{L}(F_0)}, \|P_+\|_{\mathcal{L}(F_0)}\}$. As $P_+(F_0)$ is finite dimensional, we may also assume that

$$\|P_+v\|_{F_\nu} = \|P_+v\|_{F_0}, \quad v \in F_0, \quad \nu \in \{\mu, 1\},$$

where we also use the fact that $P_+F_0 \subset D(L^n)$ for every $n \in \mathbb{N}$, c.f. [30, Proposition A.1.2].

CLAIM 1: For any initial value $\tilde{\rho}_0 \in \mathbb{B}_{F_\mu}(0, \delta)$, $P_+\tilde{\rho}$ admits the representation

$$(5.7) \quad P_+\tilde{\rho}(t) = - \int_t^\infty e^{L_+(t-s)} P_+g(\tilde{\rho}(s)) ds \quad t \geq 0.$$

For this we first establish that, for $\tilde{\rho}_0 \in \mathbb{B}_{F_\mu}(0, \delta)$,

$$e^{-\omega t} \tilde{\rho} \in BC_{1-\mu}(\mathbb{R}_+, F_1) := \left\{ u \in C((0, \infty), F_1) : \sup_{t \in \mathbb{R}_+} t^{1-\mu} \|u(t)\|_{F_1} < \infty \right\}.$$

First notice that the mapping property

$$g : \mathbb{F}_1(J_T) \cap C(J_T, U_0) \rightarrow \mathbb{F}_0(J_T), \quad 0 < T < \infty,$$

which follows in the same way as the mapping property derived for \mathcal{G}_\star in the proof of Theorem 4.9 above, together with the inequalities (5.2) and (5.4) yield

$$(5.8) \quad \begin{aligned} & \|e^{-\omega t} P_-\tilde{\rho}\|_{B_{1-\mu}(J_T, F_1)} \\ & \leq M_1 \left(\|P_-\tilde{\rho}_0\|_{F_\mu} + C_0\beta \|e^{-\omega t} P_+\tilde{\rho}\|_{B_{1-\mu}(J_T, F_1)} + C_0\beta \|e^{-\omega t} P_-\tilde{\rho}\|_{B_{1-\mu}(J_T, F_1)} \right) \end{aligned}$$

for any $0 < T < \infty$. Due to (5.6), we have $M_1C_0\beta \leq 1/2$ and can further conclude

$$(5.9) \quad \|e^{-\omega t} P_-\tilde{\rho}\|_{B_{1-\mu}(J_T, F_1)} \leq 2M_1 \left(\|P_-\tilde{\rho}_0\|_{F_\mu} + C_0\beta \|e^{-\omega t} P_+\tilde{\rho}\|_{B_{1-\mu}(J_T, F_1)} \right).$$

It follows from (5.5) that

$$t^{1-\mu} \|e^{-\omega t} P_+\tilde{\rho}(t)\|_{F_1} \leq t^{1-\mu} e^{-\omega t} C_0 \|\tilde{\rho}(t)\|_{F_\mu} \leq C_0 C_1 \varepsilon$$

where $C_1 := \sup\{t^{1-\mu} e^{-\omega t} : t \geq 0\} < \infty$. Inserting this result into (5.9) yields

$$(5.10) \quad \|e^{-\omega t} \tilde{\rho}\|_{B_{1-\mu}(J_T, F_1)} \leq 2M_1 \|P_-\tilde{\rho}_0\|_{F_\mu} + (2M_1 C_0\beta + 1) C_0 C_1 \varepsilon \leq C_2$$

for any $0 < T < \infty$. However, since T is arbitrary and (5.10) is independent of T we conclude that $e^{-\omega t} \tilde{\rho} \in BC_{1-\mu}(\mathbb{R}_+, F_1)$, for any initial value $\tilde{\rho}_0 \in \mathbb{B}_{F_\mu}(0, \delta)$. Next we note that, for $s \geq t$, by (5.3)

$$(5.11) \quad \begin{aligned} \|e^{L_+(t-s)} P_+g(\tilde{\rho}(s))\|_{F_0} & \leq M_0 C_0 \beta e^{(\omega+\gamma)(t-s)} \|\tilde{\rho}(s)\|_{F_1} \\ & \leq M_0 C_0 \beta e^{\omega t} e^{\gamma(t-s)} s^{\mu-1} \|e^{-\omega s} \tilde{\rho}\|_{B_{1-\mu}(\mathbb{R}_+, F_1)}, \end{aligned}$$

which shows that the integral in (5.7) exists for any $t \geq 0$, with convergence in F_1 . Moreover,

$$(5.12) \quad \left\| \int_t^\infty e^{L_+(t-s)} P_+ g(\tilde{\rho}(s)) ds \right\|_{F_0} \leq e^{\omega t} M_0 C_0 C_3 \beta \|e^{-\omega t} \tilde{\rho}\|_{B_{1-\mu}(\mathbb{R}_+, F_1)},$$

where $C_3 := \sup \left\{ \int_t^\infty e^{\gamma(t-s)} s^{\mu-1} ds : t \geq 0 \right\} < \infty$. Noting that $w = P_+ \tilde{\rho}$ solves the Cauchy problem

$$\begin{cases} \dot{w} - L_+ w = P_+ g(\tilde{\rho}), \\ w(0) = P_+ \tilde{\rho}_0, \end{cases}$$

it follows from the variation of parameters formula that, for $t \geq 0$ and $\tau > 0$,

$$P_+ \tilde{\rho}(t) = e^{L_+(t-\tau)} P_+ \tilde{\rho}(\tau) + \int_\tau^t e^{L_+(t-s)} P_+ g(\tilde{\rho}(s)) ds.$$

Since this representation holds for any $\tau > 0$, the claim follows from (5.3) and (5.5) by sending τ to ∞ .

CLAIM 2: If $\tilde{\rho}_0 \in \mathbb{B}_{F_\mu}(0, \delta)$ and $\|\tilde{\rho}(t, \tilde{\rho}_0)\|_{F_\mu} < \varepsilon$ for all $t \geq 0$, then it must hold that

$$\|P_+ \tilde{\rho}_0\|_{F_\mu} \leq 2M_0 M_1 C_3 \|P_- \tilde{\rho}_0\|_{F_\mu}.$$

From (5.7) and (5.11) follows

$$(5.13) \quad \begin{aligned} & \|e^{-\omega t} P_+ \tilde{\rho}\|_{B_{1-\mu}(\mathbb{R}_+, F_0)} \\ & \leq \frac{M_0 C_0 \beta}{\gamma} \left(\|e^{-\omega t} P_+ \tilde{\rho}\|_{B_{1-\mu}(\mathbb{R}_+, F_1)} + \|e^{-\omega t} P_- \tilde{\rho}\|_{B_{1-\mu}(\mathbb{R}_+, F_1)} \right) \end{aligned}$$

where we have used the fact that $\sup_{t \geq 0} \left\{ t^{1-\mu} \int_t^\infty e^{\gamma(t-s)} s^{\mu-1} ds \right\} \leq 1/\gamma$. Adding the estimates in (5.8) and (5.13) and employing (5.6) yields

$$(5.14) \quad \|e^{-\omega t} \tilde{\rho}\|_{B_{1-\mu}(\mathbb{R}_+, F_1)} \leq 2M_1 \|P_- \tilde{\rho}_0\|_{F_\mu}.$$

The representation (5.7) in conjunction with (5.12) and (5.14) then implies

$$(5.15) \quad \|P_+ \tilde{\rho}_0\|_{F_\mu} \leq M_0 C_0 C_3 \beta \|e^{-\omega t} \tilde{\rho}\|_{B_{1-\mu}(\mathbb{R}_+, F_1)} \leq M_0 C_3 \|P_- \tilde{\rho}_0\|_{F_\mu},$$

where the last inequality follows from the fact that $2C_0 M_1 \beta \leq 1$. We have thus demonstrated the claim.

Notice that the preceding claim contradicts the stability assumption. In particular, if $\tilde{\rho}_0 \in \mathbb{B}_{F_\mu}(0, \delta)$ is chosen such that $P_- \tilde{\rho}_0 = 0$, then it must hold that $P_+ \tilde{\rho}_0 = 0$, and hence $\tilde{\rho}_0 = 0$, which contradicts the assumption of stability for arbitrary $\tilde{\rho}_0 \in \mathbb{B}_{F_\mu}(0, \delta)$. \square

We also state the following corollary, which establishes instability of small cylinders for the original problem (1.2). The corollary is easily proved by use of the projection P_0 , which serves as a connection between the problems (1.2) and (4.5), and application of the result established in the theorem above. In fact, the same techniques used to prove Theorem 5.1 can also be employed to prove the corollary directly.

Corollary 5.2. *Let $r_* \in (0, 1)$ and $\mu \in [1/2, 1]$ be fixed with $4\mu + \alpha \notin \mathbb{Z}$. Then the equilibrium r_* of (1.2) is unstable in the topology of $h^{4\mu+\alpha}(\mathbb{T})$ for initial values in $h^{4\mu+\alpha}(\mathbb{T})$.*

6. BIFURCATION RESULTS

In this section we turn our attention back to the general equilibria of (1.2). In particular, we are interested in the interactions between the family of cylinders and the family of unduloids. We have already seen that the radius $r_\star = 1$ plays a critical role in the dynamics of the cylinders. The change of stability for cylinders above and below this critical radius suggests that there is a bifurcation at $r_\star = 1$. Indeed, we will confirm this bifurcation, using results of Crandall and Rabinowitz [13], and investigate properties of the bifurcation. Herein we take the parameter $\lambda := 1/r_\star$ as our bifurcation parameter, $r_\star > 0$.

With the tools and reductions developed in Section 4, we see that it suffices to study the bifurcation equation

$$(6.1) \quad \bar{\mathcal{G}}(\tilde{\rho}, \lambda) := \mathcal{G}_\star(\tilde{\rho}, 0) = P_0 G(\psi(\tilde{\rho}) + r_\star) = 0, \quad \lambda = 1/r_\star,$$

in the setting of $(\tilde{\rho}, \lambda) \in F_1 \times (0, \infty)$, where we use $\psi(\tilde{\rho}) := \psi(\tilde{\rho}, 0)$ to economize notation. Recalling the explicit characterization (4.17), we note that the eigenvalues of $D_1 \mathcal{G}_\star(0, 0)$ all have multiplicity two in the setting of F_1 , regardless of the value of the parameter r_\star . From this observation we see that the techniques of Crandall and Rabinowitz [13], where the authors derive results for operators with simple eigenvalues, are not directly applicable in this setting. We may choose at this point to employ more general bifurcation results for high dimensional kernels, such as the results contained in Kielhöfer [27, Section I.19], or we can simplify the setting in which we are working in order to make accessible the results of [13].

Whether we choose to simplify our current setting or use the higher dimensional bifurcation results, we can make good use of the following observation. Due to the periodicity enforced in the problem, the set of equilibria of (1.2) is invariant under shifts along the axis of rotation. More precisely, recalling the translation operators T_a as discussed in the proof of Theorem 2.4, one can easily verify that $G(T_a \bar{r}) = 0$ if and only if $G(\bar{r}) = 0$, $a \in \mathbb{R}$. Obviously, this invariance carries over to the reduced problem (4.5) and subsequently to the bifurcation equation (6.1).

One can take advantage of this shift invariance of equilibria in the context of bifurcation with high dimensional kernels by constructing a two dimensional bifurcation parameter $\tilde{\lambda} = (1/r_\star, a)$ and eventually observe two dimensional bifurcating *surfaces* of equilibria in F_1 , c.f. [27, Theorem I.19.2 and Remarks I.19.3]. On the other hand, we will make use of this invariance to simplify the setting in which we are looking for equilibria and make accessible the methods of Crandall and Rabinowitz for operators with simple eigenvalues. The specific simplification that we apply to our setting has also been employed by Escher and Matioc [19] and is supported by the following proposition which allows us to consider the class

$$F_{1,e} := h_{0,e}^{4+\alpha}(\mathbb{T})$$

of functions which are even, i.e. symmetric about $[x = 0]$, and $h_0^{4+\alpha}$ regular.

Proposition 6.1. *For every equilibrium $\bar{\rho}$ of (4.5), there exists $x_0 = x_0(\bar{\rho}) \in \mathbb{T}$ for which the translation $T_{x_0} \bar{\rho}$ is in the space $F_{1,e} := h_{0,e}^{4+\alpha}(\mathbb{T})$ of even functions on \mathbb{T} in the class F_1 . I.e. up to translations on \mathbb{T} , all equilibria of (4.5) are even functions.*

Proof. From Remarks 3.2 and Proposition 4.6, we know that $\bar{\rho}$ must correspond with the projection of an undulary curve $R(\mathcal{H}, B)$, modulo translations along the

x -axis. Choose $x_0 \in \mathbb{T}$ so that $T_{x_0}\bar{\rho} = P_0 R(\cdot; \mathcal{H}, B)$ and one readily verifies that $R(\cdot; \mathcal{H}, B)$ is symmetric about $s = \pi/2\mathcal{H}$. The claim now follows from $x(\pi/2\mathcal{H}) = 0$. \square

From this observation, we see that there is no loss of generality if we focus our bifurcation analysis on the setting of $\bar{\rho} \in F_{1,e}$. One benefit of working in this setting is that we have the Fourier series representation

$$\bar{\rho}(x) = \sum_{k \geq 1} a_k \cos(kx), \quad \{a_k\} \subset \mathbb{R} \quad \text{for all } \bar{\rho} \in F_{1,e}.$$

We are now prepared to prove our first bifurcation result.

Theorem 6.2 (Bifurcation of Reduced Problem). *For every $\ell \in \mathbb{N}$, $(0, \ell) \in h_{0,e}^{4+\alpha}(\mathbb{T}) \times (0, \infty)$ is a bifurcation point for the equation (6.1). In particular, there exists a positive constant $\delta_\ell > 0$ and a nontrivial analytic curve*

$$(6.2) \quad \{(\bar{\rho}_\ell(s), \lambda_\ell(s)) \in h_{0,e}^{4+\alpha} \times \mathbb{R} : s \in (-\delta_\ell, \delta_\ell), (\bar{\rho}_\ell(0), \lambda_\ell(0)) = (0, \ell)\},$$

such that

$$\bar{\mathcal{G}}(\bar{\rho}_\ell(s), \lambda_\ell(s)) = 0 \quad \text{for all } s \in (-\delta_\ell, \delta_\ell),$$

and all solutions of (6.1) in a neighborhood of $(0, \ell)$ are either a trivial solution $(0, \lambda)$ or an element of the nontrivial curve (6.2). Moreover, if $\lambda \in (0, \infty) \setminus \mathbb{N}$, then $(0, \lambda)$ is not a bifurcation point for (6.1).

Proof. We first note that bifurcation can only occur at points $(0, \lambda)$ for which $D_1 \bar{\mathcal{G}}(0, \lambda)$ is not bijective. We can see from (4.14)–(4.15) that

$$(6.3) \quad D_1 \bar{\mathcal{G}}(0, \lambda) = -\partial_x^2 (\lambda^2 + \partial_x^2) \Big|_{F_{1,e}},$$

which is realized as a Fourier multiplier with the symbol

$$(M_k)_{k \in \mathbb{N}} = (k^2(\lambda^2 - k^2))_{k \in \mathbb{N}},$$

and we see that the operator is bijective whenever $\lambda \in (0, \infty) \setminus \mathbb{N}$. Hence, it follows that bifurcation can only occur at points of the form $(0, \ell)$, $\ell \in \mathbb{N}$.

Now fix $\ell \in \mathbb{N}$ and we proceed to verify that $(0, \ell)$ is indeed a bifurcation point for (6.1). By compactness of the resolvent $R(\lambda) := (\lambda - DG_\star(0))^{-1}$, $\lambda \in \rho(DG_\star(0))$, it follows that $D_1 \bar{\mathcal{G}}(0, \ell)$ is a Fredholm operator of index zero. Further, we see that

$$N_\ell := N(D_1 \bar{\mathcal{G}}(0, \ell)) = \text{span}\{\cos(\ell x)\},$$

$$R_\ell := R(D_1 \bar{\mathcal{G}}(0, \ell)) = \overline{\text{span}}\{\cos(kx) : k \geq 1, k \neq \ell\},$$

where $N(B)$ and $R(B)$ denote the kernel and the range, respectively, of the operator B . Since $h^\sigma(\mathbb{T}) \hookrightarrow L_2(\mathbb{T})$, we can borrow the L_2 -inner product to realize N_ℓ as a topological complement to R_ℓ as subspaces of $F_{1,e}$. Meanwhile, following from (6.3), we compute the mixed derivative

$$(6.4) \quad D_2 D_1 \bar{\mathcal{G}}(0, \ell) = -2\ell \partial_x^2 \Big|_{F_{1,e}}.$$

Now take $\hat{v}_0 := \cos(\ell \cdot) \in N_\ell$ and observe that

$$D_2 D_1 \bar{\mathcal{G}}(0, \ell) \hat{v}_0 = 2\ell^3 \cos(\ell \cdot) \notin R_\ell,$$

from which the result follows by [13, Theorem 1.7], or [27, Theorem I.5.1]. \square

Remark 6.3. Following from the previous result, we are able to track the behavior of the so-called *critical eigenvalue* $\mu_\ell(\lambda)$ of the linearization $D_1\bar{\mathcal{G}}(0, \lambda)$ about the trivial equilibria $(0, \lambda)$. In particular, we choose $\mu_\ell(\lambda)$ to be the eigenvalue of $D_1\bar{\mathcal{G}}(0, \lambda)$ which passes through 0 with non-vanishing speed at $\lambda = \ell$, the existence of $\mu_\ell(\lambda)$ is guaranteed by the bifurcation observed above, c.f. [27, Section I.6 and I.7]. Moreover, employing eigenvalue perturbation techniques, we can also track the associated perturbed eigenvalue $\hat{\mu}_\ell(s)$ of the linearization $D_1\bar{\mathcal{G}}(\bar{\rho}_\ell(s), \lambda_\ell(s))$ about the nontrivial equilibria. These eigenvalues will play a crucial role in the following instability results for the branches of bifurcating equilibria.

Theorem 6.4. *Each of the bifurcations established in Theorem 6.2 is a subcritical pitchfork type bifurcation. More precisely, for all $\ell \in \mathbb{N}$, we have*

$$\dot{\lambda}_\ell(0) = 0 \quad \text{and} \quad \ddot{\lambda}_\ell(0) < 0,$$

where “ \cdot ” denotes the derivative with respect to the parameter s . Moreover, it holds that the perturbed eigenvalues $\hat{\mu}_\ell(s)$ are strictly positive for $|s| > 0$ chosen sufficiently small.

Proof. Utilizing the methods of [27, Section I.6 and I.7], and the techniques developed in the previous sections of the paper, one can explicitly verify that the bifurcations observed above are indeed subcritical pitchfork bifurcations. To begin, for $\ell \in \mathbb{N}$ fixed, recall the subspaces R_ℓ , N_ℓ of $F_{1,e}$, and $\hat{v}_0 = \cos(\ell \cdot)$ as defined in the proof of Theorem 6.2 above. First, one computes the second derivative

$$D_1^2\bar{\mathcal{G}}(0, \ell)[\hat{v}_0, \hat{v}_0] = -6\ell^5 \cos(2\ell \cdot) \in R_\ell,$$

from which we conclude $\dot{\lambda}_\ell(0) = 0$. Meanwhile, utilizing the representation [27, (I.6.11) and (I.6.9)] and following a considerable amount of computation, one will see that

$$\ddot{\lambda}_\ell(0) = -\frac{15\ell^3}{6} < 0.$$

From these observations, we conclude that the bifurcation observed at the point $(0, \ell)$ is a subcritical pitchfork bifurcation.

The result for the perturbed eigenvalues now follows from the eigenvalue perturbation techniques in [27, Section I.7], see also Amann [1, Section 27]. \square

With these bifurcation results established in the setting of the reduced problem, we will now go about deriving results for the original problem (1.2). Recalling the definition of the operator G_\star from Section 4.1, we introduce the notation

$$G(\rho, \lambda) := G(\rho + 1/\lambda) = G_\star(\rho), \quad \text{for } \lambda = 1/r_\star.$$

We are now interested in finding solutions to the bifurcation equation

$$(6.5) \quad G(\rho, \lambda) = 0, \quad (\rho, \lambda) \in h^{4+\alpha}(\mathbb{T}) \times (0, \infty),$$

associated with the full problem (1.2).

We begin analyzing (6.5) by *lifting* the bifurcation results already established for the reduced problem. We make use of the connections established in Section 4.2 and we also establish the following connection between the eigenvalues of $D_1\bar{\mathcal{G}}$ and $DG(\cdot, \lambda)$ at equilibria.

Proposition 6.5. *Suppose $\bar{\mathcal{G}}(\bar{\rho}, \lambda) = 0$ and $\mu \neq 0$. Then*

$$\mu \text{ is an eigenvalue for } D_1\bar{\mathcal{G}}(\bar{\rho}, \lambda) \iff \mu \text{ is an eigenvalue for } D_1G(\psi(\bar{\rho}), \lambda).$$

Proof. (i) First, suppose that $D_1\bar{\mathcal{G}}(\tilde{\rho}, \lambda)\tilde{h} = \mu\tilde{h}$ for some $\tilde{h} \in F_1 \setminus \{0\}$, and let $h := D\psi(\tilde{\rho})\tilde{h}$. Then $h \in E_1 \setminus \{0\}$, by injectivity of $D\psi(\tilde{\rho})$, and it follows from (4.13) that

$$D_1G(\psi(\tilde{\rho}), \lambda)h = \mu h.$$

We also observe that this assertion is true in case $\mu = 0$.

(ii) Now suppose that $D_1G(\psi(\tilde{\rho}), \lambda)h = \mu h$ for some $h \in E_1 \setminus \{0\}$. We conclude from (4.12) that $h \in T_{\psi(\tilde{\rho})}\mathcal{M}_0$, so that there exists a unique $\tilde{h} \in F_1 \setminus \{0\}$ for which $h = D\psi(\tilde{\rho})\tilde{h}$. Then (4.13) shows that

$$\mu D\psi(\tilde{\rho})\tilde{h} = D\psi(\tilde{\rho})D_1\bar{\mathcal{G}}(\tilde{\rho}, \lambda)\tilde{h},$$

and finally, by injectivity of $D\psi(\tilde{\rho})$, we conclude that $\mu\tilde{h} = \bar{\mathcal{G}}(\tilde{\rho}, \lambda)\tilde{h}$, as desired. \square

We are now prepared to prove the main result regarding bifurcation of the original problem (1.2) in the setting of $h^{4+\alpha}(\mathbb{T})$, and instability of the bifurcating unduloids.

Theorem 6.6 (Bifurcation of Full Problem). *Fix $\ell \in \mathbb{N}$. Then:*

a) *the set*

$$(6.6) \quad \left\{ \psi(\tilde{\rho}_\ell(s)) + 1/\lambda_\ell(s) : s \in (-\delta_\ell, \delta_\ell) \right\} \subset h^{4+\alpha}(\mathbb{T}),$$

is an analytic curve of equilibria for the problem (1.2) which bifurcates subcritically from the family of cylinders $r_\star \in (0, \infty)$, at the cylinder $r_\star = 1/\ell$.

b) *there exists some $\varepsilon_\ell > 0$ so that for every $s \in (-\delta_\ell, \delta_\ell)$*

$$\psi(\tilde{\rho}_\ell(s)) + 1/\lambda_\ell(s) = R(B, \ell), \quad \text{for some } B \in (-\varepsilon_\ell, \varepsilon_\ell),$$

i.e. the family (6.6) of equilibria are exactly the even presentations of $2\pi/\ell$ -periodic undulary curves in some neighborhood of the cylinder $r_\star = 1/\ell$.

c) *the undulary curves $R(B, \ell)$ are unstable for $|B| > 0$ chosen sufficiently small.*

Proof. (a) It follows from Proposition 4.6 and Theorem 6.2 that the family

$$\left\{ (\psi(\tilde{\rho}_\ell(s)), \lambda_\ell(s)) : s \in (-\delta_\ell, \delta_\ell) \right\} \subset E_1 \times (0, \infty)$$

consists of solutions to the bifurcation equation (6.5). The regularity of the curve follows from the regularity of the bifurcating branch in Theorem 6.2 and regularity of the mapping ψ . By definition of the bifurcation function $G(\cdot, \lambda)$, it follows that the family (6.6) are indeed equilibria of the original equation (1.2) which intersect the family of cylinders at $r_\star = 1/\ell$, when $s = 0$. Meanwhile, the bifurcation parameter λ remains unchanged in lifting from the reduced problem to the full problem, hence we see that

$$\dot{\lambda}_\ell(0) = 0 \quad \text{and} \quad \ddot{\lambda}_\ell(0) < 0,$$

from Theorem 6.4, and so we conclude that the given curve bifurcates subcritically.

(b) By Remarks 4.1(f) it follows that ψ preserves the symmetry of even functions on \mathbb{T} , and since $\tilde{\rho}_\ell(s) \in F_{1,e}$, it follows that the functions in the family (6.6) are even on \mathbb{T} . Meanwhile, by the characterization of equilibria established in Section 3, and the fact that

$$\psi(\tilde{\rho}_\ell(0)) + 1/\lambda_\ell(0) = 1/\ell = R(0, \ell),$$

it follows that the family (6.6) must coincide with the family of $2\pi/\ell$ -periodic undulary curves $R(B, \ell)$, for some continuum of values $B \in (-\varepsilon_\ell, \varepsilon_\ell)$.

(c) To prove that the unduloids (6.6) are unstable, we mimic the proof of Theorem 5.1 in the current setting. In particular, define

$$\begin{aligned} G_\ell(\rho, s) &:= G(\rho + \psi(\tilde{\rho}_\ell(s)), \lambda_\ell(s)), \quad \text{and} \\ L_\ell(s) &:= D_1 G_\ell(0, s) = D_1 G(\psi(\tilde{\rho}_\ell(s)), \lambda_\ell(s)), \end{aligned}$$

acting on functions $\rho \in E_1$. It follows by Theorem 6.4 and Proposition 6.5 that

$$\sigma(L_\ell(s)) \cap [\operatorname{Re} z > 0] \neq \emptyset,$$

provided $|s| > 0$ is chosen sufficiently small. Meanwhile, the operator $G_\ell(\cdot, s)$ has a similar quasilinear structure as \mathcal{G}_\star and so the analogue to inequality (5.2) is also derived for

$$g_\ell(\rho, s) := G_\ell(\rho, s) - L_\ell(s)\rho.$$

Utilizing [27, Proposition I.7.2] and the explicit characterization (4.16) of the spectra $\sigma(DG_\star(\eta))$, we can control the eigenvalues of the perturbed linearization $L_\ell(s)$, so that, for sufficiently small values of $|s| > 0$, we can derive the necessary *spectral gap* condition

$$[\omega - \gamma \leq \operatorname{Re} z \leq \omega + \gamma] \cap \sigma(L_\ell(s)) = \emptyset \quad \text{and} \quad \sigma_+ := [\operatorname{Re} z > \omega + \gamma] \cap \sigma(L_\ell(s)) \neq \emptyset,$$

for some $\gamma, \omega > 0$. The remainder of the proof now follows as in the proof of Theorem 5.1 with the observation that $-L_\ell(s)$ satisfies maximal regularity properties, which follows by uniform ellipticity of $L_\ell(s)$ and an argument similar to the proof of Claim 1 in Lemma 2.1. \square

Remark 6.7. The instability result Theorem 6.6(c) can also be proved in the reduced setting for the functions $\tilde{\rho}_\ell(s)$, again provided that $|s| > 0$ is sufficiently small. However, it is more difficult to prove that the associated linearization $L_\ell(s)$ satisfies maximal regularity properties due to the presence of the zero-mean projection P_0 in the definition of the operators in the reduced setting.

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